

ON RISK-AVERSE AND ROBUST INVENTORY PROBLEMS

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To my wife,

Pelin,

and to my parents,

Nursel and Kemal

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TABLE OF CONTENTS

DEDICATION	iii
ACKNOWLEDGEMENTS	iv
LIST OF TABLES	vii
LIST OF FIGURES	viii
SUMMARY	ix
CHAPTERS	
I INTRODUCTION	1
1.1 Motivation	1
1.2 Objectives	8
1.3 Organization	9
II BACKGROUND	11
2.1 Classical stochastic inventory theory	11
2.2 Risk averse models	14
2.3 Min-max and robust models	17
2.4 Coherent risk measures	22
III COHERENT RISK MEASURES IN SINGLE-PERIOD NEWSVENDOR PROBLEMS	26
3.1 Introduction	26
3.2 Single-period newsvendor models	26
3.3 Optimal solution structure	27
3.4 Effect of risk aversion on optimal order quantity	30
3.5 Numerical illustration	35
IV COHERENT RISK MAPPINGS IN MULTI-PERIOD INVENTORY PROBLEMS	38
4.1 Introduction	38
4.2 Multi-period newsvendor models	38

4.3	Optimal policy structure	41
4.4	The multi-period problem with setup cost	44
V	ROBUST INVENTORY PROBLEMS: INDEPENDENT UNCERTAINTY SETS	
	CASE	47
5.1	Introduction	47
5.2	Dynamic robust inventory problems	47
5.3	Static models	50
5.4	Optimal solution structure	54
5.5	Fixed ordering cost case	62
VI	ROBUST INVENTORY PROBLEMS: DEPENDENT UNCERTAINTY SETS	
	CASE	64
6.1	Introduction	64
6.2	Dynamic robust model with dependent uncertainty sets	64
6.3	Optimality of base-stock policy	66
6.4	Budget of uncertainty approach	67
6.5	Proposed heuristic solution	69
6.6	Alternative models and solutions	72
6.7	Computational results	76
VII	CONCLUSION AND FUTURE RESEARCH	78
7.1	Conclusion	78
7.2	Future research	80
	REFERENCES	82
	VITA	87

LIST OF TABLES

1	Optimal order quantity	36
2	Comparison of solutions	76
3	Comparison of solutions	77

LIST OF FIGURES

1	Cumulative distribution of cost for both solutions	5
2	Maximum, minimum and average cost for different mean values	37
3	Maximum, minimum and average cost for different standard deviation values	37

SUMMARY

This thesis focuses on the analysis of various extensions of the classical multi-period single-item inventory problem. Specifically, we investigate two particular approaches of modeling risk in the context of inventory management, risk averse models and robust formulations.

We analyze the classical newsvendor problem with linear cost terms utilizing a coherent risk measure as our objective function. Properties of coherent risk measures allow us to offer a unifying treatment of risk averse and min-max type formulations. For the single period newsvendor problem, we show that the structure of the optimal solution of the risk averse model is similar to that of the classical expected value problem. For a finite horizon dynamic inventory model, we show that, again, the optimal policy has a similar structure as that of the expected value problem. This result carries over even to the case when there is a fixed ordering cost. We also provide conditions on the optimality of myopic policies and analyze monotonicity properties of the optimal order quantity with respect to the degree of risk aversion for certain risk measures.

We expand our analysis to robust formulations of multi-period inventory problem. We consider independent uncertainty sets and prove the optimality of base-stock policies and the computational tractability of some specific cases. We obtain results for fixed ordering cost case similar to our previous formulations. We generalize our model and formulate the dynamic robust model for dependent uncertainty sets. We prove that base-stock policy is optimal even for the most generalized problem formulation. We focus on budget of uncertainty approach and develop a heuristic that can also be employed for a class of parametric dependency structures. We compare our proposed heuristic against alternative solution techniques.

CHAPTER I

INTRODUCTION

1.1 Motivation

Basic inventory problems are about planning production or deciding ordering quantities to build an inventory so as to satisfy demand. Keeping an inventory to meet demand on time is essential for many companies. In general, retailers are the most important sector facing inventory problems. However, both manufacturing and service industries have similar problems in different settings. The purpose of inventory theory is to provide these companies rules and policies for efficient management of their inventory. In a simple inventory system there is a supply process and demand process. Supply process adds stock, hence builds the inventory. On the other hand, demand process consumes the inventory. During these activities, depending on the system, various costs such as setup cost, ordering cost, unit cost for the item, holding cost, and back ordering cost are incurred. These cost terms add up to a significant amount for most companies and this is why their management is crucial.

In modern market conditions, customers do not tolerate frequent shortages caused by low inventory levels. Besides lost sales, shortages may destroy a company's reputation and market share. However, keeping high inventory is not the answer either since it is not always possible in practice and may be very costly. In the last financial crisis business world witnessed that during times of global recession, the access to capital may be very limited and expensive. Financial cost of the working capital tied up to high inventory levels will hurt the profit margins. These two conflicting factors, avoiding shortages and lowering capital tied up to inventory, are the basis for the significance of inventory theory.

In the classical textbook by Zipkin [73], it is reported that as of March 1999, the inventory carried by businesses in the United States worth about \$ 1.1 trillion, which is about 1.35 times their total monthly sales. Obviously, this huge investment should be planned wisely. The same work relates the success of big companies like Wal-Mart, Toyota and Dell to their ability to efficiently manage their inventories. In a more recent work, Chen *et al.* [21] studied the inventory levels of publicly traded American manufacturing companies between 1981 and 2000, trying to examine the relationship between good inventory management and profitability. They discovered that firms with higher than normal inventory levels have poor long-term stock returns and firms with lowest inventory levels have only ordinary returns. Moreover, they observed that the firms with slightly lower than average inventories have good stock returns. These works suggest that, to be successful a company needs policies that minimize its inventory related costs while ensuring customer satisfaction through product availability. These policies can be obtained only by building and solving realistic inventory models.

The earliest inventory models assume that demand is a known deterministic process. These deterministic models laid the foundations of the modern inventory theory, however, in order to obtain models that are more representative of real life, the uncertain nature of the demand process should be taken into consideration. In today's volatile markets it is not even possible to determine the exact distribution of demand let alone its exact value for a given period. The detailed analysis of models with explicit stochastic features started in 1950's. The pioneering works of Arrow *et al.* [4] and Dvoretzky *et al.* [30] were followed by many others. Most of the basic techniques used in the area such as dynamic programming and stationary analysis were established in this period. These techniques provided most of the important results in this area and are still used in current works.

Usage of stochastic demand processes produced more realistic models and at the same time gave birth to different modeling approaches. In the classical modeling approach one assumes that the distribution of demand is known. With this knowledge it is possible to

calculate the expected value of inventory related costs for any given policy. The classical newsvendor model is based on optimizing this expected value.

This thesis is concerned with the single-item inventory problem where the inventory planner tries to decide an ordering quantity, or in the multi-period case an ordering policy, in an attempt to efficiently manage inventory related costs. The basic inventory problem that we consider is the well-known newsvendor problem, originally analyzed by Within [71]. Despite being very simplistic, this problem is studied extensively in inventory management literature and still receives attention because of its increasing relevance due to shortening product life cycle. The main reason behind the significance of this model is the existence of many practical applications that share the same simple structure. These applications include retailers who need to determine stock levels (multi-period dynamic inventory planning), manufacturers who need to determine operating capacity (capacity planning) and hotels and airlines who need to manage their booking policies (revenue management). In fact, many airline revenue management systems employ heuristics developed by Belobaba [7, 8, 9] utilizing the results from Littlewood's work [51] based on models similar to the newsvendor problem. For the details of this connection the reader is referred to Phillips [56] and Talluri and van Ryzin [68].

Next, we provide a simple version of the newsvendor problem for demonstration of different modeling approaches. A newsvendor who does not have any newspaper at the beginning, must decide on its inventory level x . He will pay c dollars per unit, hence the order cost is cx . After the products arrive, which is assumed to happen immediately when the order is placed, demand is realized. We denote the random demand by D and a specific realization by d . According to the realized demand d the newsvendor will pay either a holding cost h for every unit left in inventory or a back-ordering cost b for every unit of unsatisfied demand. Hence the total cost of newsvendor for a specific realization d is

$$C(x, d) = cx + h[x - d]_+ + b[d - x]_+. \quad (1)$$

We assume that the objective is to minimize the expected total cost. When we have

complete knowledge of the distribution of demand it is possible to calculate the expected cost for the newsvendor. Suppose F is the cumulative distribution function for demand, then the expected total cost for a given inventory level x is $\mathbb{E}_F[C(x, D)]$. Utilizing this problem definition and notation the classical newsvendor model can be formulated as

$$\text{Min}_{x \in \mathbb{R}^+} \mathbb{E}_F[C(x, D)]. \quad (2)$$

One weakness of the classical newsvendor model is the risk-neutral setting assumption. In a risk-neutral setting, one disregards how $C(x, D)$ is distributed for a given x and focuses only on its expected value. In certain cases, this may be a good approach. For example, if you have exactly the same problem for a large number of different items and if you are interested only in the total cost, then you can use risk-neutral setting because whatever the cost distribution is for a specific item, the total cost will converge to some fixed number by Law of Large Numbers. However, in this case newsvendor will face the cost corresponding to a specific realization. Hence, one should also consider different values that $C(x, D)$ may take for different realizations. In risk-neutral setting this variability is not addressed. Consider a problem where you have to choose one of two simple games offering two outcomes with equal probabilities. In the first game you either gain 0 or 100 dollars and in the second game you either gain 40 or 60 dollars. According to risk-neutral setting there is no difference between these two games. In reality, some people will choose the first game and some will choose the second game because of different risk preferences. Schweitzer and Cachon [65] provide experimental evidence suggesting that for some products, labeled as high profit items, inventory planners are risk-averse. It is very natural to assume that some inventory managers would be willing to accept higher expected cost as a price for protection against extreme cases and variability. The following example shows that risk-neutral setting is not satisfactory for these inventory managers.

Example 1 *Consider the model given in (2). Let $c = 10$, $b = 12$ and $h = 4$ and assume that the demand is distributed normally with mean 70 and standard deviation 20. The optimal*

inventory level for this problem is $x^* = 47$, the resulting expected cost is 765.2. We generate 10000 demand values using the assumed distribution. For $x^* = 47$ the average of highest 1000 cost values is 1165.44. If we use $x = 55$ the expected cost is 771.57, which is not very high compared to minimum value. For the same generated demand values the average of highest 1000 cost is 1149.43. To summarize, the expected value increases by 0.8 percent, whereas the average of highest values decreases by 1.4 percent. This may be desirable for some inventory managers and undesirable for others.

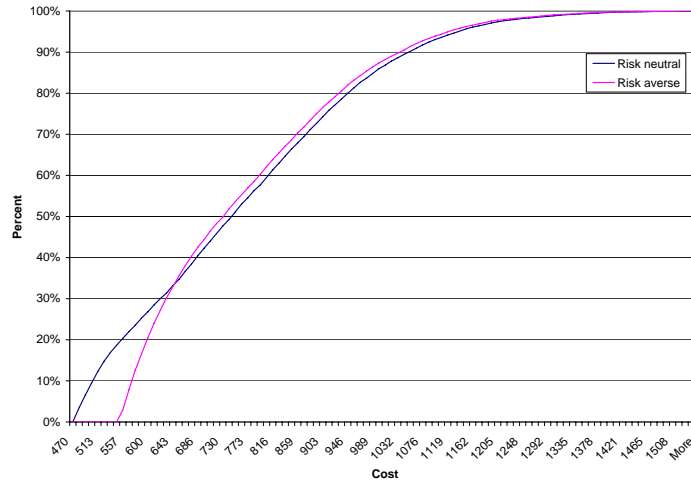


Figure 1: Cumulative distribution of cost for both solutions

Figure 1 shows that if newsvendor uses risk-averse ordering quantity $x = 55$ instead of risk-neutral one, then the probability that his cost is less than or equal to 765.2 gets bigger. In fact this argument is valid for any cost value greater than 680.

Another important drawback of the classical model is the assumption of complete knowledge of the distribution of demand. In real life the exact distribution of demand is almost never available. In some cases, it may be possible to derive a distribution with the help of historical demand data and expert opinion. However, there is no easy way to examine the effects of using this derived distribution instead of exact one. In today's continuously changing market conditions, companies are not content about utilizing distributions derived from historical data without even knowing the impact of estimation errors.

These two shortcomings of classical inventory models resulted in two separate streams of research. Researchers in the first stream try to incorporate risk into their models through use of different objective functions. The risk-averse models are the product of this approach. The works on the other stream focus on eliminating unrealistic assumption on the availability of complete information about the underlying random demand process and are called min-max or robust models.

Risk-averse models are very similar to classical models in risk-neutral setting. The difference between them is the objective function. Instead of expected value of total cost, risk-averse models use different objective functions that take the variability of total cost into consideration. One particular replacement is the expected value of a utility function U of total cost. Consider the problem formulated in (2) for risk-neutral setting. If we choose to use a utility function for a risk-averse model, we obtain

$$\text{Min}_{x \in \mathbb{R}^+} \mathbb{E}_F[U(C(x, D))]. \quad (3)$$

The use of an expected utility objective is the most widely applied technique to address risk aversion [50, 31, 2, 49]. Another option that received some attention is the use of a mean-variance criterion [20, 28]. This idea was introduced by Markowitz [52, 53] in the context of financial models and it is still widely used in finance, both in theory and in practice. This approach leads to the following model

$$\text{Min}_{x \in \mathbb{R}^+} \mathbb{E}_F[C(x, D)] + \lambda \text{Var}_F(C(x, D)), \quad (4)$$

where Var_F denotes the variance of the total cost with respect to cumulative distribution function F and λ is a coefficient representing the risk preference of the decision maker. Conditional value-at-risk (CVaR), another risk measure originally employed in financial models by Rockafellar and Uryasev [57, 59], appeared in some recent works on risk-averse inventory models [41, 55]. Finally, a specific group of risk measures, namely law-invariant coherent risk measures, are recently utilized in the context of newsvendor model both for single and multi-product cases [26, 27].

All of these works will be discussed in detail in Section 2.2. Here we would like to state some disadvantages of these models. Risk-averse models assume that complete knowledge of demand distribution is available, as in classical inventory models. Moreover, in multi-period case most of these models are solvable only within the practical limitations of dynamic programming. Besides these common problems different choices for risk aversion may create different issues. For example, utility functions are conceptual and they do not provide much insight to the decision maker. On the other hand, mean-variance criterion is criticized for punishing both sides of variability, although low cost values are desirable. Moreover, Ahmed [3] proves that this approach leads to non-convex optimization problems and computational intractability.

The min-max or robust approach tries to deal with the issue regarding imprecision in the underlying distribution. Even if we do not know the distribution of demand exactly, often we can reasonably identify a relevant family \mathcal{A} of probability distributions. Then one way to formulate the problem is to minimize the worst-case expected cost over this family of distributions. A min-max type model for the classical newsvendor formulation given in (2) is

$$\underset{x \in \mathbb{R}^+}{\text{Min}} \underset{F \in \mathcal{A}}{\text{Max}} \mathbb{E}_F[C(x, D)]. \quad (5)$$

Some works using min-max models assume partial information about demand distribution [63, 39], whereas others assume nothing about the distribution besides some bounds on maximum and minimum demand values [47, 48]. Recently, researchers studied supply chain management problems, including simple inventory problems, using robust optimization framework [17, 18]. Robust optimization does not always result in a min-max type model because the focus is on feasibility. However, in the problems that we consider the feasibility is not an issue and robust models are equivalent to min-max type formulations.

Min-max and robust models have their own drawbacks. They are in general criticized for being over-conservative by considering only some extreme cases that are highly improbable in reality and for their inability of representing the risk preferences of the decision

maker. Furthermore, their static nature prevents them capturing the true system dynamics in multi-period problems. Recently developed models eliminating the static nature are either based on approximations or fail to provide the same computational advantages of the classical models.

Risk-averse and min-max models developed quite separately from each other. They have different advantages and setbacks. To our knowledge there is no detailed work trying to compare and relate these two approaches.

1.2 Objectives

In this thesis, we use coherent risk measures to control variability in the context of inventory problems. The notion of coherent risk measures is introduced recently by Artzner *et al.* [5] and since then received great attention in risk-management literature. We will present the definition, properties and examples of coherent risk measures in Section 2.4. Examples of such risk measures are conditional value-at-risk and mean-absolute semideviation, which, being based on dispersion statistics, avoid the use of hard-to-elicited utility functions for modeling risk aversion. Unlike the classical mean-variance criterion, coherent risk measures satisfy stochastic dominance conditions and result in convex optimization problems.

An important property of coherent risk measures allows us to build a connection and obtain a one-to-one correspondence between the risk-averse and min-max type models. Hence, we offer a unified treatment of two streams of research mentioned above. We extend this analysis to multi-period inventory problem and relate risk-averse formulation to nested min-max formulation. For both single-period and multi-period problems, we analyze the optimal ordering policy structure. Even for the case of fixed ordering cost we prove that very well known policies are optimal for risk averse models. Moreover, we investigate monotonic behavior for the optimal ordering quantity for single-period problem and existence of myopic optimal policies for the multi-period problem.

Then, we examine static and dynamic robust formulations of multi-period inventory problem. We demonstrate the advantages of dynamic robust formulation and analyze the optimal policy structure for the case where the uncertainty sets for demand in different periods are assumed to be independent. Using the results of our work on risk averse models, we obtain the optimal policy structure for a very general dynamic robust model. It turns out that when linear holding and backordering costs are assumed, the dynamic robust model is computationally tractable. Moreover, we provide a closed form solution scheme to dynamic robust model under some assumptions. We also present results for fixed ordering cost case parallel to our previous ones.

We extend our analysis to the case of dependent uncertainty sets and show that many of our results are valid for this case as well. We try to analyze a specific dependency relation and suggest a heuristic solution approach for the dynamic robust model valid for a class of parametric dependency structure including the specific one we consider. We present computational results comparing our suggested heuristic method to other alternatives.

1.3 Organization

The organization of this thesis is as follows. Chapter 2 provides background information about classical inventory models and coherent risk measures. We also review literature on risk-averse and min-max type inventory models in the same chapter. In Chapter 3, we consider using coherent risk measures in single-period newsvendor problem and analyze the resulting models. This approach is extended to multi-period inventory problems in Chapter 4. Chapter 5 focuses on the analysis of static and dynamic robust models and development of solution techniques for the dynamic robust model when the uncertainty sets are independent of each other. We consider the generalized version of dynamic robust models in Chapter 6 by allowing the uncertainty sets to have some sort of dependency structure. For a specific dependency structure, we propose a heuristic solution methodology and compare it with alternative techniques. Finally, in Chapter 7 we summarize the main

contributions of the thesis and propose future research directions.

CHAPTER II

BACKGROUND

Modern inventory theory started with the derivation of economic order quantity (EOQ) formula by Harris [44]. Over the next few decades many different variations were studied. Significant portion of these assumed a deterministic demand process. In the early 1950's, works that consider stochastic demand processes emerged. Arrow *et al.* [4] and Dvoretzky *et al.* [30], both considered demand as a random variable with known distribution and used a risk neutral setting. Many researcher followed their footsteps over the following decade. The classical text of Hadley and Whitin [43] is a comprehensive summary of these early developments. The involvement of stochastic processes led to different modeling approaches. Here we review works employing these different approaches. Lastly, we try to give detailed information about coherent risk measures which will be utilized to build a connection between separate methodologies.

2.1 Classical stochastic inventory theory

Early works on classical stochastic inventory models established the foundation of modern inventory theory. A variety of inventory models are extensively studied under this stream of research which produced very well-known results. An important portion of these results are about the structure of optimal policy of numerous inventory problems in risk neutral setting. Studies investigating various models and optimal solution structures include Scarf [64], Clark and Scarf [29], Iglehart [45], Bellman *et al.* [6], Veinott [69, 70], Ehrhart [32] and Federgruen and Zipkin [35]. The optimal policy structures examined in these works still receives attention and many researchers continue to extend the results from these early studies. Zipkin [73] provides a complete coverage of the ideas and findings in this area. Here we discuss some of the classical models and their optimal solutions in detail.

We have already presented a simple version of the classical newsvendor problem in Section 1.1 without providing much detail. Note that for the model (2) to be meaningful, one needs to assume that $b > c$ holds. Otherwise the optimal solution would be not to order and back-order the whole demand. Instead of this assumption on problem parameters we may modify the model so that the inventory left at the end of the period is taken into account. Similar small variations lead to many alternative models.

Different variants of the newsvendor model can be obtained by assuming that there is a fixed ordering cost and/or by taking into account the income newsvendor will receive from the sales and trying to maximize the profit. In the latter, unit selling price may be a fixed parameter or a decision variable which also affects the distribution of the demand. Another possible variation can be obtained by utilizing nonlinear holding and/or backordering cost functions.

Consider the problem of a newsvendor who is trying to decide his inventory level x and whose initial inventory is y . He needs to pay c dollars for each unit, but there is no fixed ordering cost. The newsvendor knows the cumulative distribution of demand F . We assume that lead-time is zero, in other words newspapers arrive at the moment newsvendor orders. According to the realized demand d and its inventory level the newsvendor will pay $h[x - d]_+ + b[d - x]_+$ as combined holding and backordering cost. At the end of period, regardless of the sign of the inventory $x - d$ there is a terminal cost of $-c_1(x - d)$. Under these assumptions the newsvendor problem is

$$\text{Min}_{x \geq y} \mathbb{E}_F[c(x - y) + h[x - D]_+ + b[D - x]_+ - c_1(x - D)]. \quad (6)$$

The optimal solution for this model is a *base-stock policy*.

Definition 1 *A base-stock policy with a base-stock level x^* is a policy where the newsvendor orders up to x^* if $x^* > y$ and does not order otherwise.*

To model the case where there is a fixed ordering cost k , we need to define an additional function. Let $\varphi(x - y)$ be equal to 1 if $x - y > 0$ and zero otherwise, so that the ordering

cost is $k\varphi(x - y)$. Then, our model becomes

$$\text{Min}_{x \geq y} \mathbb{E}_F[c(x - y) + k\varphi(x - y) + h[x - D]_+ + b[D - x]_+ - c_1(x - D)]. \quad (7)$$

The optimal solution for this new model is an (s, S) policy.

Definition 2 *In an (s, S) policy, the newsvendor orders $S - y$ if $y \leq s$ and does not order otherwise.*

We provide the definition of these policies because many variants of these models have the same optimal solution structure. Moreover, the multi-period extensions of these models again have the same optimal solution structure.

Consider a multi-period extension of the model (6) and assume that our planning horizon T is finite. We can employ the same variables and parameters by only incorporating the time index t . Assume demand is given by a random process D_1, \dots, D_T and for every $i = 2, \dots, T$, the distributions of D_i , given by the cumulative distribution function F_i , is independent of (D_1, \dots, D_{i-1}) . Of course, in multi-period case the terminal cost occurs at the end of the planning horizon and is equal to $-c_{T+1}(x_T - d_T)$. The dynamic programming formulation for this problem can be written down as follows. Let

$$V_T(y_T) = \text{Min}_{x_T \geq y_T} \mathbb{E}_{F_T}[c_T(x_T - y_T) + h_T[x_T - D_T]_+ + b_T[D_T - x_T]_+ - c_{T+1}(x_T - D_T)] \quad (8)$$

be the value function for the last period and for $t = 1, \dots, T - 1$ define

$$V_t(y_t) = \text{Min}_{x_t \geq y_t} \mathbb{E}_{F_t}[c_t(x_t - y_t) + h_t[x_t - D_t]_+ + b_t[D_t - x_t]_+ + V_{t+1}(x_t - D_t)]. \quad (9)$$

The optimal solution for these dynamic programming equations is again a base-stock policy. Moreover, it is possible to extend these equations in a straight forward way to the case where there is a fixed ordering cost. We only need to add the term $k\varphi(x_t - y_t)$ to every value function. The optimal solution for this extended model is an (s, S) policy.

Note that calculating the optimal inventory levels for single period newsvendor models is usually a simple task. On the other hand obtaining the optimal base-stock levels for

multi-period extension is not always easy. Although dynamic programming has its theoretical advantages, for complex problems its practical use is limited because of the so-called "curse of dimensionality". This term refers to the large number of variables resulting from enumeration of all possible states, which may be the only option to solve the dynamic programming equations when it is not possible to separate the value functions from each other. To overcome this limitation researchers considering multi-period models tried either using approximate models or looking for myopic optimal solutions.

2.2 *Risk averse models*

The fact that classical risk-neutral setting disregards variability, motivated researchers to incorporate risk in inventory models. One of the common techniques applied to achieve this goal is the use of utility functions. Lau [50] analyzes the single period newsvendor model under two different objective functions. First one is based on maximizing the decision maker's expected utility of total profit, whereas the second objective is to maximize the probability of achieving a target level of profit. Eeckhoudt *et al.* [31] examines the effects of risk and risk aversion in the single period newsvendor problem where an increasing and concave utility function based on profit is used. In this risk averse setting the effects of changes in various price and cost parameters are investigated. They report monotonicity properties of optimal order quantity with respect to risk aversion. Agrawal and Seshadri [2] conduct a similar study for a generalized model where the demand distribution is a function of the selling price. They consider two different ways the price decision can affect the demand distribution. They show that compared to a risk neutral decision maker, a risk averse one will order more in one setting and less in another. They discuss the implications of these results on supply chain strategy and channel design. A more recent work by Keren and Pliskin [49] analyzes the single period risk averse newsvendor problem with expected utility maximization approach. They derive the first order optimality conditions and use them to obtain a closed form solution for a special case where the demand is uniformly

distributed and the utility function is increasing and differentiable.

Utility functions are also employed in multi-period models. Bouakiz and Sobel [19] show the optimality of base-stock policy for the multi-period newsvendor problem optimized with respect to an exponential utility criterion. The utility function they consider is based on negative of present value of costs. In the same work infinite horizon problem is also examined and it is demonstrated that the optimal infinite horizon policy is ultimately stationary. Chen *et al.* [22] provide similar results for more general models. They consider all combinations of two modeling assumptions; whether the price is a decision variable or not, and whether there is a fixed ordering cost or not. They also incorporate hedging opportunities into their models and analyze the resulting formulation. It is reported that in many cases risk averse models share the same optimal solution structure with the risk neutral one.

Other techniques of addressing risk aversion issue includes using a mean-variance criterion or a risk measure such as conditional value-at-risk. Chen and Federgruen [20] as well as Choi *et al.* [28] analyze mean-variance trade-offs in different inventory problems, including newsvendor model. The first study [20] demonstrates that when mean-variance approach is considered, the relationship between the optimal ordering quantity for risk averse model and risk neutral one depends on whether a profit or cost based model is utilized. The same work also examines two infinite horizon models; one with the objective of minimizing the expected steady state costs and another one considering a disutility function based on the mean and variance of the customer waiting time. Gotoh and Takano [41] use conditional value-at-risk (CVaR) in the context of single period newsvendor problem. They show that utilizing downside risk measures including the CVaR results in tractable problems and provide analytical solutions for the minimization of CVaR measures defined with two different loss functions. A recent work by Özler *et al.* [55] utilize value-at-risk (VaR) as the risk measure in a newsvendor framework and investigate multi-product version under a VaR constraint. They derive the exact distribution function for the two-product

newsvendor problem and develop an approximation method for the profit distribution of the N -product case. Their approach is suitable to handle a wide range of cases including the correlated demand case.

Coherent risk measures are a class of risk functions satisfying certain conditions and this specific class recently received some attention from researchers looking for different methods of handling risk in the context of inventory management. Choi and Ruszczyński [26] consider law-invariant coherent risk measures and derive an equivalent representation of a risk-averse newsvendor problem as a mean-risk model. This model is utilized to obtain results on monotonicity of order quantity with respect to risk aversion. Choi *et al.* [27] extend this work to multi-product case. They provide closed form approximations of optimal order quantities when the demands are independent and demonstrate that when the number of products goes to infinity the risk neutral solution is asymptotically optimal. The effects of positively or negatively dependent demands on optimal order quantities are also investigated.

An important issue concerning these formulations is the unrealistic assumption that the complete knowledge of demand distribution is given. In many real life problems this information is not available. When there is historical data one can calculate statistics such as mean and standard deviation but it may not be possible to fit the data to a distribution. Another common problem is practical limitations for multi-period extensions. Obtaining the actual solution for these type of models depends on solving the dynamic programming equations. This task proves to be a hard one when the planning horizon T is large.

There are also issues related to specific techniques. Utility functions are too conceptual to identify and they do not provide much insight. On the other hand mean-variance criterion penalizes even the desirable outcomes, leads to non-convex optimization problems in multi-period problems and do not satisfy stochastic dominance conditions.

2.3 *Min-max and robust models*

The efforts to eliminate the requirement for complete knowledge of demand distribution resulted in min-max type inventory models. When the whole information on the distribution of random factors is not available, one way to formulate a problem involving stochastic elements is to assume that the actual distribution of these elements lies in a given family of distributions and try to optimize the worst possible value of the objective function. The first such work in the context of inventory problems is due to Scarf [63], who considers a single period newsvendor problem and determines the ordering quantities that maximize the minimum expected profit over all possible continuous demand distributions with a given mean and variance. Another early work by Kasugai and Kasegai [47] proposes a dynamic programming approach to the distribution free multi-period newsvendor problem when the only available information about the demand is a known closed interval that covers all possible realizations. Same authors [48] also examined the minimax regret ordering principle under the same problem setting and compared it with their previous study.

Decades later Gallego and Moon [39] provide a brief derivation of Scarf's results for single period newsvendor problem and consider various extensions of the problem. Gallego [37] extends minimax approach to the infinite horizon continuous review (Q, R) inventory model with incidence oriented backorder costs. He also analyzes the same problem with time weighted backorder costs [38]. Moon and Gallego [54] apply similar techniques to the infinite horizon continuous and periodic review models with backorder and lost sales. In a more recent work, Gallego *et al.* [40] consider multi-period stochastic inventory problems with discrete demand distributions. They assume that the available information on demand distributions is limited to selected moments, percentiles, or a combination of moments and percentiles. They show that when the objective is to minimize the maximum expected cost, many inventory models of this form can be solved by a sequence of linear programs. Another recent work by Yue *et al.* [72] considers min-max type models for several holding and backordering cost functions. For a general cost function they identify the favorable and

unfavorable demand distributions and provide bounds for the cost function.

A tool that is widely used in the last decade to formulate min-max type models is robust optimization. In robust optimization framework random factors are considered as uncertain parameters that lie in a given set and only decisions feasible for every possible choice of parameters in this uncertainty set are considered during optimization. The idea goes back to Soyster [67], however the classical references by Ben-Tal and Nemirovski [12, 13, 14] provide the foundation of the area. In these works a group of convex optimization problems with uncertain parameters is studied. These problems are formulated as conic programs which can be solved in polynomial time. Specifically, [13] considers the linear programming (LP) problems with uncertain data. The robust counterpart for an uncertain LP is formulated and analytical and computational optimization tools are developed. It is proven that the robust counterpart of an LP with an ellipsoidal uncertainty set is a conic quadratic program, a computationally tractable problem. Ben-Tal and Nemirovski [14] study 90 LPs from the well known Net Lib collection and apply robust optimization techniques. They show that in many cases the optimal solution of an LP becomes infeasible as a result of small perturbations of the problem data and the optimal solution of robust counterpart stays feasible without giving up a lot in optimality. Around the same period, El Ghaoui and Lebret [33] consider least-squares problem where the coefficient matrices are subject to uncertainty and try to minimize the worst case residual error. They show that under some assumptions on the perturbations this problem can be solved in polynomial time using semidefinite programming. In [34] these results are extended to semidefinite programs whose data depend on some unknown but bounded perturbation parameters. The techniques presented in these works were applied to many different fields including inventory management.

Bertsimas and Thiele [17] study robust formulations of several multi-period inventory problems including a general version of the newsvendor problem. They utilize the budget of uncertainty approach to model the demand uncertainty sets. This specific approach is

based on limiting the total variation from the expected values and is developed by Bertsimas and Sim [16] as an alternative to interval based uncertainty sets that are criticized for being too conservative and to computationally demanding ellipsoidal uncertainty sets. Although ellipsoidal uncertainty sets lead to polynomially solvable problems, these problems require more computational effort than a similar size linear program. Bertsimas and Sim [16] show that when budget of uncertainty approach is employed, robust counterpart of an LP is also a linear optimization problem. To take advantage of this approach Bertsimas and Thiele [17] consider a linear approximation of multi-period single item inventory problem (the original version of this problem is non-convex) and formulate the robust counterpart. They analyze the optimal solution structure of the robust model and extend their analysis to the multi-echelon problem. Using a similar approach, Adida and Perakis [1] suggest a deterministic robust optimization formulation to address demand uncertainty in a dynamic pricing and inventory control problem for a make-to-stock manufacturing environment. Bienstock and Özbay [18] consider the original non-convex robust multi-period inventory model and the case where only a constant base-stock policy is considered feasible. They analyze two different models for the demand uncertainty sets; one is the budget of uncertainty approach and the second is the bursty demand model based on empirical data. They present a Benders' decomposition based generic algorithm for solving this type of problems. Note that all of these works assume that all decisions are fixed at the beginning of the planning horizon, in other words static robust models are considered. In fact this is not the case for most real life multi-period problems. In a setting where random events occur in every period, the decision maker will have the ability to adapt his decision for a given period after observing the realizations of random factors before that specific period.

To deal with the static nature of robust models, Iyengar [46] proposes a robust formulation for discrete time dynamic programming. He models the uncertainty by associating a set of conditional measures with each state-action pair and shows that under certain conditions on this set of measures all the main results for finite and infinite horizon dynamic

programming extend to natural robust counterparts. To address the same issue, Ben-Tal *et al.* [11] introduce the notion of adjustable robust formulations. In this approach the decision variables are divided into two groups; adjustable and non-adjustable decisions. The latter are the ones that should be fixed from the beginning of the planning horizon. On the other hand adjustable decision variables refer to decisions that can be delayed until some future period and determined after observing some of the random factor realizations. In adjustable robust formulation, adjustable decision variables are expressed as a function of random factor realizations. Ben-Tal *et al.* [11] consider employing the idea with linear programming problems, however the adjustable robust counterpart is usually computationally intractable. To overcome this, the authors suggest using an affine control mechanism defining the relationship between the adjustable decisions and random factor realizations. The resulting affinely adjustable robust counterpart is solvable in polynomial time in certain important cases and have a computationally tractable approximation in other cases. This technique is applied to a retailer supplier contract problem by Ben-Tal *et al.* [10]. The affinely adjustable robust counterpart of this problem turns out to be a conic quadratic problem, known to be polynomially solvable. A large simulation study suggests that the methodology provides excellent results. Chen and Zhang [25] introduce the extended affinely adjustable robust counterpart for modeling and solving multiperiod uncertain linear programs with fixed recourse. Their work is based on reparameterization of uncertainties and applying affinely adjustable robust methodology. This approach provides deterministic and tractable optimization formulations. In a recent work, Bertsimas *et al.* [15] prove that for a specific class of one dimensional constrained multi-period robust optimization problem affine control policies are optimal. They consider a finite horizon problem with min-max type objective where the state costs are convex and control costs are linear. Note that the dynamic version of the linear approximation of multi-period single item inventory problem utilized by Bertsimas and Thiele [17] is a problem that falls under the class examined by Bertsimas *et al.* [15]. Hence affinely adjustable robust counterpart provides the optimal

policy for this problem.

In an effort to combine different ideas from robust optimization and stochastic models, Chen *et al.* [23] introduced a methodology to construct uncertainty sets utilizing the notion of directional deviations labeled as forward and backward deviations. The authors consider a class of multi-stage chance constrained stochastic linear problems and propose a tractable approximation based on linear decision rule. This work is extended by Chen *et al.* [24] with the analysis of several piecewise linear decision rules. In light of these new ideas, See and Sim [66] considered applying adjustable robust modeling techniques in the context of multi-period inventory management. Instead of an affine control mechanism they use a piecewise linear one and show that this approach also leads to a tractable optimization problem in the form of second order cone optimization problem. They represent demand ambiguity utilizing a factor based model. It is assumed that the available information on demand distribution is limited to mean, support and some measures of deviations. The authors consider a newsvendor who is trying to minimize maximum expected cost for all distributions possessing these characteristics. An upper bound for the expected cost is obtained using the information on demand distribution and the model is developed to minimize this upper bound. Static, affine and piecewise linear replenishment policies are analyzed. Computational results suggests that proposed piecewise linear replenishment policies perform better than static and affine policies and other policies obtained using simple heuristics derived from dynamic programming, such as a myopic policy.

Min-max type models eliminates the assumption of complete knowledge of the distribution of underlying factors. Moreover, these models are often solvable or have such an approximation. Despite these advantages, they are often seen as too conservative. This is because these models consider only highly unlikely extreme cases. A solution that gives desirable outcomes for almost all realization of random factors may be eliminated by these models because it performs really bad for few realizations, even if the probability of these

realizations is very close to zero. In addition, these models do not have the ability to represent the risk preference of the inventory manager, since their approach to variability is not very different from risk-neutral approach.

2.4 Coherent risk measures

Since coherent risk measures are an essential part of our work we would like to give detailed information and present some available literature about them. The notion of coherent risk measures arose from an axiomatic approach for quantifying the *risk* of a financial position. Artzner *et al.* [5] first define "acceptable" future random net values and provide a group of axioms about the set of acceptable future net values. Then, they state axioms on measures of risk and relate them to the axioms on acceptance sets.

Definition 3 Consider a random outcome Z viewed as an element of a linear space \mathcal{Z} of measurable functions, defined on an appropriate sample space. According to [5], a risk measure is a mapping from \mathcal{Z} into \mathbb{R} . A risk measure $\rho : \mathcal{Z} \rightarrow \mathbb{R}$ is said to be coherent if it satisfies the following axioms:

- A1. *Convexity:* $\rho(\alpha Z_1 + (1 - \alpha)Z_2) \leq \alpha\rho(Z_1) + (1 - \alpha)\rho(Z_2)$ for all $Z_1, Z_2 \in \mathcal{Z}$ and all $\alpha \in [0, 1]$.
- A2. *Monotonicity:* If $Z_1, Z_2 \in \mathcal{Z}$ and $Z_2 \geq Z_1$, then $\rho(Z_2) \geq \rho(Z_1)$.
- A3. *Translation Equivariance:* If $a \in \mathbb{R}$ and $Z \in \mathcal{Z}$, then $\rho(Z + a) = \rho(Z) + a$.
- A4. *Positive Homogeneity:* If $\alpha > 0$ and $Z \in \mathcal{Z}$, then $\rho(\alpha Z) = \alpha\rho(Z)$.

(The notation $Z_2 \geq Z_1$ means that $Z_2(\omega) \geq Z_1(\omega)$ for all elements ω of the corresponding sample space.)

Note that the original definition of coherent risk measures includes subadditivity instead of convexity, however subadditivity is equivalent to convexity when A4 holds.

Relevance and significance of the axioms A1-A4 are by now well-established in the risk-management literature. Artzner *et al.* [5] also try to justify these four desirable properties. Monotonicity is a necessary condition to prevent accumulation of risk to remain undetected. Translation equivariance is a very natural requirement implying that the effect of a guaranteed amount to the total outcome of a random event should be exactly equal to this amount. Subadditivity is based on the theory which states that diversification cannot create extra risk. Notice that subadditivity implies $\rho(\alpha Z) \leq \alpha \rho(Z)$ for any integer α . The reverse inequality is required to model what a government or exchange might impose in a situation where no netting or diversification occurs. Then the equality is generalized for any positive α . Instead of subadditivity we use convexity in our description due to the importance of convexity properties in the context of optimization.

Two particular examples of coherent risk measures, which will be discussed in more details later, are the mean-absolute deviation

$$\rho_\lambda[Z] := \mathbb{E}_F[Z] + \lambda \mathbb{E}_F|Z - \mathbb{E}_F[Z]|, \quad (10)$$

and the conditional-value-at-risk

$$CVaR_\alpha[Z] := \inf_{t \in \mathbb{R}} \left\{ t + (1 - \alpha)^{-1} \mathbb{E}_F[Z - t]_+ \right\}. \quad (11)$$

In the above examples F is a reference probability distribution, $\lambda \in [0, 1/2]$ and $\alpha \in (0, 1)$ are the corresponding parameters and Z has a finite mean $\mathbb{E}_F[Z]$.

Beside the basic properties and important examples, we would like to present the following result from [5].

Theorem 1 *With every coherent risk measure $\rho : \mathcal{Z} \rightarrow \mathbb{R}$ is associated a (convex) set \mathcal{A} of probability measures, in the dual space to \mathcal{Z} , such that the following dual representation holds*

$$\rho[Z] = \sup_{F \in \mathcal{A}} \mathbb{E}_F[Z]. \quad (12)$$

Conversely, for every convex set \mathcal{A} of probability measures such that the right-hand-side of (12) is real-valued, the corresponding function ρ is a coherent risk measure.

This important duality property allows us to relate risk-averse and min-max type models.

As the idea of coherent risk measures is relatively new, most of the research conducted so far is theoretical. After Artzner *et al.* [5] introduced the concept of coherent risk measures, the idea was developed by Föllmer and Schied [36], Rockafellar *et al.* [58], Ruszczyński and Shapiro [62] and others. In the last work mentioned a general duality framework is developed. Later Ruszczyński and Shapiro [61] introduce an axiomatic definition of a conditional risk mapping, derive its properties and develop dynamic programming relations for multi-stage optimization problems involving conditional risk mappings. Same authors examine various risk functions and establish a framework for the usage of risk measures in multi-period optimization problems in [60].

In order to build dynamic models of risk, one needs to extend the concept of a risk function. This necessity produced the conditional risk mappings. The following definition is based on [60]. Let (Ω, \mathcal{F}_2) be a measurable space, \mathcal{F}_1 be a sigma subalgebra of \mathcal{F}_2 , and \mathcal{Z}_i , for $i = 1, 2$, be linear spaces of \mathcal{F}_i measurable functions $Z : \Omega \rightarrow \mathbb{R}$. Assume that $\mathcal{Z}_1 \subset \mathcal{Z}_2$ and each of these is large enough to include all \mathcal{F}_i measurable step functions. Moreover, assume that for each \mathcal{Z}_i there is a corresponding dual space \mathcal{Z}_i^* of finite signed measures on (Ω, \mathcal{F}_i) .

Definition 4 *Under the setting described above, the mapping $\rho : \mathcal{Z}_2 \rightarrow \mathcal{Z}_1$ is a conditional risk mapping if it satisfies the following axioms:*

M1. Convexity: If $\alpha \in [0, 1]$ and $Z_1, Z_2 \in \mathcal{Z}_2$, then

$$\rho(\alpha Z_1 + (1 - \alpha)Z_2) \leq \alpha \rho(Z_1) + (1 - \alpha)\rho(Z_2).$$

M2. Monotonicity: If $Z_1, Z_2 \in \mathcal{Z}$ and $Z_2 \geq Z_1$, then $\rho(Z_2) \geq \rho(Z_1)$.

M3. Translation Equivariance: If $Y \in \mathcal{Z}_1$ and $Z \in \mathcal{Z}_2$, then $\rho(Z + Y) = \rho(Z) + Y$.

The inequalities used in the first two axioms imply componentwise inequality for all elements of the corresponding sample space. To emphasize that ρ is associated with \mathcal{Z}_1 and \mathcal{Z}_2 we write it as $\rho_{\mathcal{Z}_2|\mathcal{Z}_1}$. Conditional risk mappings in a sense replace the conditional expectation concept used in multi-period dynamic programs. Without the conditional risk mappings, it is not possible to write down the dynamic programming equations. Consider the space $\mathcal{Z} := \mathcal{Z}_1 \times \dots \times \mathcal{Z}_T$ and $Z := (Z_1, \dots, Z_T) \in \mathcal{Z}$. Define the function $\tilde{\rho} : \mathcal{Z} \rightarrow \mathbb{R}$ as follows

$$\tilde{\rho}(Z) := Z_1 + \rho_{\mathcal{Z}_2|\mathcal{Z}_1}(Z_2 + \dots + \rho_{\mathcal{Z}_{T-1}|\mathcal{Z}_{T-2}}(Z_{T-1} + \rho_{\mathcal{Z}_T|\mathcal{Z}_{T-1}}(Z_T))). \quad (13)$$

The using axiom M3 iteratively one can obtain that

$$\tilde{\rho}(Z) = \rho_T(Z_1 + \dots + Z_T) \quad (14)$$

where ρ_T is the composite mapping $\rho_{\mathcal{Z}_2|\mathcal{Z}_1} \circ \dots \circ \rho_{\mathcal{Z}_T|\mathcal{Z}_{T-1}}$. This relationship between the cumulative and nested formulations is essential for the modeling of multi-period problems.

CHAPTER III

COHERENT RISK MEASURES IN SINGLE-PERIOD NEWSVENDOR PROBLEMS

3.1 Introduction

In this chapter, we try to give a unifying treatment of risk averse and min-max inventory models. We start by building and analyzing a single period newsvendor model where the objective is to minimize a coherent risk measure in Section 3.2. The analysis of single period model shows that there is a one-to-one correspondence between the min-max type models and risk averse models when risk aversion is obtained through the use of coherent risk measures. We provide the optimal solution for this model in Section 3.3. The optimal solution structure turns out to be the same as that of the classical newsvendor model. In other words a base-stock policy is optimal. Then, properties of optimal ordering quantity is analyzed and its monotonic behavior with respect to degree of risk aversion for certain risk measures is reported in Section 3.4. We show that for coherent risk measures of a particular form, the optimal order quantity increases with risk aversion. Section 3.5 is dedicated to the presentation of some numerical results .

3.2 Single-period newsvendor models

Consider the classical newsvendor problem in a cost minimization setting. The newsvendor has to decide an order quantity x so as to satisfy uncertain demand d . The cost of ordering is $c_0 \geq 0$ per unit. Once demand d is realized, if the demand exceeds order, i.e., $d \geq x$, a back order penalty of $b \geq 0$ per unit is incurred. On the other hand, if $d \leq x$ then a holding cost of $h \geq 0$ per unit is incurred. The remaining inventory $x - d$ incurs a (discounted) cost of $-\gamma c_1(x - d)$, where $c_1 \geq 0$ and $\gamma \in (0, 1]$ are salvage value and discount parameters

respectively. The total cost is then

$$c_0x - \gamma c_1(x - d) + b[d - x]_+ + h[x - d]_+ = cx + \Psi(x, d),$$

where $[a]_+ := \max\{a, 0\}$, $c := c_0 - \gamma c_1$, and

$$\Psi(x, d) := \gamma c_1 d + b[d - x]_+ + h[x - d]_+. \quad (15)$$

Note that the function $\Psi(x, d)$, and hence the cost function, are convex in x for any d . In the subsequent analysis we view the uncertain demand as a random variable, denoted D , to distinguish it from its particular realization d .

In the risk neutral setting the corresponding optimization problem is formulated as minimization of the expected value of the total cost with respect to the probability distribution of the demand D , say given by cumulative distribution function $F(\cdot)$. That is (cf., [73, section 9.4.1]),

$$\text{Min}_{x \in \mathbb{R}} \mathbb{E}_F [cx + \Psi(x, D)]. \quad (16)$$

Let us emphasize that in the above formulation (16) the optimization is performed on *average* and it is assumed that the distribution F of the demand is *known*. Let us consider the following risk averse formulation of the newsvendor problem:

$$\text{Min}_{x \in \mathbb{R}} \rho [cx + \Psi(x, D)]. \quad (17)$$

Here $\rho[Z]$ is a coherent risk measure corresponding to a random outcome Z . By using the dual representation (12) of ρ we can write problem (17) in the following min-max form:

$$\text{Min}_{x \in \mathbb{R}} \sup_{F \in \mathcal{A}} \mathbb{E}_F [cx + \Psi(x, D)]. \quad (18)$$

Thus with ρ and \mathcal{A} appropriately chosen, there is a one-to-one correspondence between risk averse (17) and min-max (18) formulations of the newsvendor problem.

3.3 Optimal solution structure

In the following we show that the risk-averse problem (17), and equivalently the min-max problem (18), has an optimal solution structurally similar to that of the classical newsvendor

problem (16). We assume that the reference cdf, denoted F^* , is such that $F^*(t) = 0$ for any $t < 0$. Similarly, it is assumed that any $F \in \mathcal{A}$ is also like that, i.e., $F(t) = 0$ for any $t < 0$. These assumptions simply eliminate distributions where negative demand is possible. We also assume that $b \geq c$ and $b + h > 0$ to avoid trivial solutions.

Theorem 2 *With any coherent risk measure ρ is associated cdf \bar{F} , depending on ρ and $\beta := (b + \gamma c_1)/(b + h)$, such that $\bar{F}(t) = 0$ for any $t < 0$, and the objective function $\psi(x) := \rho[cx + \Psi(x, D)]$ of the newsvendor problem can be written in the form*

$$\psi(x) = (b + \gamma c_1)\rho[D] + (c - b)x + (b + h) \int_{-\infty}^x \bar{F}(t)dt. \quad (19)$$

Proof. Recall that by the dual representation (12) we have that

$$\psi(x) = \sup_{F \in \mathcal{A}} \mathbb{E}_F [cx + \Psi(x, D)]. \quad (20)$$

Using integration by parts to evaluate $\mathbb{E}_F [\Psi(x, D)]$, we can write $\psi(x) = (c-b)x + (b+h)g(x)$, where

$$g(x) := \sup_{F \in \mathcal{A}} \left\{ \beta \mathbb{E}_F [D] + \int_{-\infty}^x F(t)dt \right\} \quad (21)$$

(recall that $\beta := \frac{b+\gamma c_1}{b+h}$). Since $F(\cdot)$ is a monotonically nondecreasing function, we have that $x \mapsto \int_{-\infty}^x F(t)dt$ is a convex function. It follows that the function $g(x)$ is given by the maximum of convex functions and hence is convex. Moreover, $g(x) \geq 0$ and

$$g(x) \leq \beta \sup_{F \in \mathcal{A}} \mathbb{E}_F [D] + [x]_+ = \beta \rho[D] + [x]_+, \quad (22)$$

and hence $g(x)$ is finite valued for any $x \in \mathbb{R}$. Also for any $F \in \mathcal{A}$ and $t < 0$ we have that $F(t) = 0$, and hence $g(x) = \beta \sup_{F \in \mathcal{A}} \mathbb{E}_F [D] = \beta \rho[D]$ for any $x < 0$.

Consider the right hand side derivative of $g(x)$:

$$g^+(x) := \lim_{t \downarrow 0} \frac{g(x+t) - g(x)}{t},$$

and define $\bar{F}(\cdot) := g^+(\cdot)$. Since $g(x)$ is convex, its right hand side derivative $g^+(x)$ exists, is finite and for any $x \geq 0$ and $a < 0$,

$$g(x) = g(a) + \int_a^x g^+(t)dt = \beta \rho[D] + \int_{-\infty}^x \bar{F}(t)dt. \quad (23)$$

Note that definition of the function $g(\cdot)$, and hence $\bar{F}(\cdot)$, involves the constant β and set \mathcal{A} only. Let us also observe that the right hand side derivative $g^+(x)$, of a real valued convex function, is monotonically nondecreasing and right side continuous. Moreover, $g^+(x) = 0$ for $x < 0$ since $g(x)$ is constant for $x < 0$. We also have that $g^+(x)$ tends to one as $x \rightarrow +\infty$. Indeed, since $g^+(x)$ is monotonically nondecreasing it tends to a limit, denoted r , as $x \rightarrow +\infty$. We have then that $g(x)/x \rightarrow r$ as $x \rightarrow +\infty$. It follows from (22) that $r \leq 1$, and by (21) that for any $F \in \mathcal{A}$,

$$\liminf_{x \rightarrow +\infty} \frac{g(x)}{x} \geq \liminf_{x \rightarrow +\infty} \frac{1}{x} \int_{-\infty}^x F(t) dt \geq 1,$$

and hence $r \geq 1$. It follows that $r = 1$.

We obtain that $\bar{F}(\cdot) = g^+(\cdot)$ is a cumulative distribution function of some probability distribution and the representation (19) holds. \square

Consider the number

$$\kappa := \frac{b - c}{b + h}. \quad (24)$$

Recall that it was assumed that $b + h > 0$. Therefore, $\kappa \geq 0$ iff $b \geq c (= c_0 - \gamma c_1)$. Note that by (19) we have that for $x < 0$ the objective function $\psi(x)$, of the newsvendor problem, is equal to a constant (independent of x) plus the linear term $(c - b)x$. Therefore, if $b < c$, then the objective function $\psi(x)$ can be made arbitrary small by letting $x \rightarrow -\infty$. If $b = c$, i.e., $\kappa = 0$, then $\psi(x)$ is constant for $x < 0$. Now if $\kappa > 1$, i.e., $b - c > b + h$, then the objective function $\psi(x)$ can be made arbitrary small by letting $x \rightarrow +\infty$. If $\kappa = 1$ and $\bar{F}(t) < 1$ for all t , then $\psi(x)$ is monotonically decreasing as $x \rightarrow +\infty$. Therefore, situations where $b \leq c$ or $h + c \leq 0$ are somewhat degenerate, and we assume that $\kappa \in (0, 1)$. Consider the corresponding quantile (also called value-at-risk) of the cdf \bar{F} :

$$VaR_\kappa(\bar{F}) = \bar{F}^{-1}(\kappa) := \inf \{t \in \mathbb{R} : \bar{F}(t) \geq \kappa\}. \quad (25)$$

Note that for $\kappa \in (0, 1)$ this quantile is well defined and finite valued. It follows from the representation (19) that if $\kappa \in (0, 1)$, then $\bar{x} := VaR_\kappa(\bar{F})$ is always an optimal solution of

the newsvendor problem (17). More precisely, we have the following result.

Corollary 1 *Suppose that $\kappa \in (0, 1)$. Then the set of optimal solutions of the newsvendor problem (17) is a nonempty closed bounded interval $[a, b]$, where $a := \text{VaR}_\kappa(\bar{F})$ and $b := \sup\{t \in \mathbb{R} : \bar{F}(t) \leq \kappa\}$. For $\kappa = 0$, any $x < 0$ is an optimal solution of the newsvendor problem (17).*

In some cases it is possible to calculate the corresponding cdf \bar{F} in a closed form. Consider the conditional-value-at-risk measure $\rho[Z] := \text{CVaR}_\alpha[Z]$ defined with respect to a reference cdf $F^*(\cdot)$. The corresponding set \mathcal{A} of probability measures is given by cumulative distribution functions $F(\cdot)$ such that $\mathbb{P}_F(S) \leq (1 - \alpha)^{-1} \mathbb{P}_{F^*}(S)$ for any Borel set $S \subset \mathbb{R}$ (here \mathbb{P}_F denotes the probability measure corresponding to cdf F). It follows that the cdf

$$\hat{F}_\alpha(t) := \min \left\{ (1 - \alpha)^{-1} F^*(t), 1 \right\}$$

belongs to the set \mathcal{A} and dominates any other cdf in \mathcal{A} . Suppose now that the parameters $b = 0$ and $c_1 = 0$. Then the function $g(x)$, defined in (21), can be written as follows

$$g(x) = \sup_{F \in \mathcal{A}} \int_{-\infty}^x F(t) dt = \int_{-\infty}^x \hat{F}_\alpha(t) dt.$$

That is, in this case $\bar{F} = \hat{F}_\alpha$. Of course, as it was discussed above, the case of $b = 0$ and $c_1 = 0$ is not very interesting since then $\kappa < 0$.

3.4 Effect of risk aversion on optimal order quantity

Consider now risk measures of the form

$$\rho_{\lambda, \mathbb{D}}[Z] = \mathbb{E}[Z] + \lambda \mathbb{D}[Z], \quad (26)$$

where \mathbb{E} is the usual expectation operator, taken with respect to a reference distribution F , \mathbb{D} is a measure of variability and λ is a nonnegative weight to trade off expectation with variability. Higher values of λ reflects a higher degree of risk aversion. A risk measure of the form (26) is called a mean-risk function. Not all variability measures \mathbb{D} and/or values

of λ result in the risk measure $\rho_{\lambda, \mathbb{D}}$ to be coherent. Consider the p -th semideviation as the variability measure

$$\delta_p[Z] := (\mathbb{E}[Z - \mathbb{E}[Z]]_+^p)^{\frac{1}{p}}. \quad (27)$$

Then ρ_{λ, δ_p} is a coherent risk measure for any $p \geq 1$ and $\lambda \in [0, 1]$. For $p = 1$ and λ changed to $\lambda/2$, the corresponding mean-absolute semideviation risk function coincides with the mean-absolute deviation risk function defined in (10). On the other hand if we use variance (or standard deviation) as the dispersion measure, then the corresponding mean-risk function typically does not satisfy the monotonicity condition, and hence is not a coherent risk measure, for any $\lambda > 0$.

In the following we investigate the behavior of optimal solutions to the risk-averse model (17), involving coherent mean-risk objectives, with respect to the risk aversion parameter λ .

Lemma 1 *Let $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$ be two convex functions and $S_i := \arg \min_{x \in \mathbb{R}} f_i(x)$, $i = 1, 2$, be their sets of minimizers. Suppose that S_1 and S_2 are nonempty, and hence are closed intervals $S_1 = [a_1, b_1]$ and $S_2 = [a_2, b_2]$. Consider function $f_\lambda(x) := (1 - \lambda)f_1(x) + \lambda f_2(x)$ and let $S_\lambda := \arg \min_{x \in \mathbb{R}} f_\lambda(x)$. Then for any $\lambda \in [0, 1]$ the following holds: (i) If $b_1 < a_2$ then the set S_λ is nonempty and monotonically nondecreasing in $\lambda \in [0, 1]$, (ii) If $b_2 < a_1$, then the set S_λ is nonempty and monotonically nonincreasing in $\lambda \in [0, 1]$, (iii) If the sets S_1 and S_2 have a nonempty intersection, then $S_\lambda = S_1 \cap S_2$ for any $\lambda \in (0, 1)$.*

Proof. We only prove the assertion (i), the other assertion (ii) is analogous. We have that for every $\lambda \in [0, 1]$ the function $f_\lambda(x)$ is convex. Note that since $f_1(x)$ and $f_2(x)$ are real valued and convex, and hence continuous, S_1 and S_2 are closed intervals. Since $f_1(x)$ is convex, it has finite left and right side derivatives, denoted $f_1^-(x)$ and $f_1^+(x)$, respectively. Since b_1 is a minimizer of $f_1(\cdot)$, we have that $f_1^-(b_1) \geq 0$. Also since a_2 is the smallest minimizer of $f_2(\cdot)$ and $a_2 > b_1$ we have that $f_2^-(b_1) > 0$. Consequently, for any $\lambda \in (0, 1)$ we have that $f_\lambda^-(b_1) > 0$, and hence $f_\lambda(x) > f_\lambda(b_1)$ for all $x < b_1$. By similar arguments

we have that $f_\lambda(x) > f_\lambda(a_2)$ for all $x > a_2$. By convexity arguments this implies that the set S_λ is nonempty and $S_\lambda \subset [b_1, a_2]$ for all $\lambda \in (0, 1)$. Note that the set S_λ is a closed interval, say $S_\lambda = [a_\lambda, b_\lambda]$. Also by similar arguments it is not difficult to show that for any $0 \leq \lambda < \lambda' \leq 1$, it holds that $b_\lambda \leq a_{\lambda'}$.

In order to prove (iii) note that for any $\lambda \in (0, 1)$, $y \in \mathbb{R}$ and $x \in S_1 \cap S_2$ we have

$$(1 - \lambda)f_1(y) + \lambda f_2(y) \geq (1 - \lambda)f_1(x) + \lambda f_2(x),$$

and that the above inequality is strict if $y \notin S_1 \cap S_2$. □

Now let us make the following observations. If $\rho_i : \mathcal{Z} \rightarrow \mathbb{R}$, $i = 1, 2$, are two coherent risk measures, then their convex combination $\rho_\lambda[Z] := (1 - \lambda)\rho_1[Z] + \lambda\rho_2[Z]$ is also a coherent risk measure for any $\lambda \in [0, 1]$. Since, for any d , the function $x \mapsto cx + \Psi(x, d)$ is convex, the functions $f_i(x) := \rho_i[cx + \Psi(x, D)]$, $i = 1, 2$, are convex real valued (convexity of the composite functions f_i follows by convexity and monotonicity of ρ_i). Therefore, by the above lemma, we have that if functions $f_1(\cdot)$ and $f_2(\cdot)$ have disjoint sets of minimizers S_1 and S_2 , respectively (recall that by Corollary 1 these sets are nonempty), then the set S_λ , of minimizers of $\rho_\lambda[cx + \Psi(x, D)]$ is monotonically nondecreasing or nonincreasing in $\lambda \in [0, 1]$, depending on whether $S_2 > S_1$ or $S_1 < S_2$.

Theorem 3 *Consider the newsvendor problem with a mean-risk objective $\rho_{\lambda, \mathbb{D}}$ of the form (26). Suppose that $\rho_{\lambda, \mathbb{D}}$ is a coherent risk measure for all $\lambda \in [0, 1]$ and let S_λ be the set of optimal solutions of the corresponding problem. Suppose that the sets S_0 and S_1 are nonempty. Then the following holds.*

- (i) *If $S_0 \cap S_1 = \emptyset$, then S_λ is monotonically nonincreasing or monotonically nondecreasing in $\lambda \in [0, 1]$ depending upon whether $S_0 > S_1$ or $S_0 < S_1$. If $S_0 \cap S_1 \neq \emptyset$, then $S_\lambda = S_0 \cap S_1$ for any $\lambda \in (0, 1)$.*

- (ii) Consider some $x_0 \in S_0$ such that $f_0(x) := cx + \mathbb{E}[\Psi(x, D)]$ is twice continuous differentiable near x_0 with $f_0''(x_0) \neq 0$ and $v(x) := \mathbb{D}[\Psi(x, D)]$ is continuously differentiable near x_0 . If $v'(x_0) > 0$ then S_λ is monotonically nonincreasing; if $v'(x_0) < 0$ then S_λ is monotonically nondecreasing; if $v'(x_0) = 0$ then $S_\lambda = \{x_0\}$ for all $\lambda \in [0, 1]$.

Proof.

- (i) Note that for any $\lambda \in [0, 1]$ the objective function of the newsvendor problem with a mean-risk objective $\rho_{\lambda, \mathbb{D}}$ is

$$f_\lambda(x) = (1 - \lambda)f_1(x) + \lambda f_2(x),$$

where $f_1(x) := cx + \mathbb{E}[\Psi(x, D)]$ and $f_2(x) := cx + \mathbb{E}[\Psi(x, D)] + \mathbb{D}[\Psi(x, D)]$. The result then follows from Lemma 1.

- (ii) Note that since f_0 is convex and $f_0''(x_0) \neq 0$, it follows that $f_0''(x_0) > 0$ and hence x_0 is the unique minimizer of $f_0(x)$. Since $v(x)$ is continuously differentiable at x_0 , we have then that for all $\lambda > 0$ small enough, a minimizer $x_\lambda \in S_\lambda$ is a solution of the equation $f_0'(x) + \lambda v'(x) = 0$. It follows by the Implicit Function Theorem that for $\lambda > 0$ small enough the minimizer x_λ is unique and

$$\frac{dx_\lambda}{d\lambda} = -\frac{v'(x_0)}{f_0''(x_0)}.$$

Combining the above with (i) the result follows. \square

Let us check the sign of $v'(x_0)$ corresponding to the p -th semideviation risk measure (27) taken with respect to cdf F . It is sufficient to check the sign of

$$\frac{d}{dx} \left(\mathbb{E}[\Psi(x, D) - \mathbb{E}[\Psi(x, D)]]_+^p \right) \Big|_{x=x_0}. \quad (28)$$

Suppose that the reference cdf $F(\cdot)$ is continuous. Then the above derivative exists and the derivative can be taken inside the expectation. Letting $\Delta(x, d) := p[\Psi(x, d) - \mathbb{E}\Psi(x, D)]_+^{p-1}$,

the derivative (28) is equal to:

$$\begin{aligned}
&= \int \Delta(x_0, t) \frac{d}{dx} [\Psi(x_0, t) - \mathbb{E}[\Psi(x_0, D)]]_+ dF(t) \\
&= \int_{\Psi(x_0, t) \geq \mathbb{E}[\Psi(x_0, D)]} \Delta(x_0, t) \frac{d}{dx} [\Psi(x_0, t) - \mathbb{E}[\Psi(x_0, D)]] dF(t) \\
&= \int_{\Psi(x_0, t) \geq \mathbb{E}[\Psi(x_0, D)]} \Delta(x_0, t) [\Psi'_x(x_0, t) + c] dF(t), \tag{29}
\end{aligned}$$

where the last line follows from the optimality conditions for x_0 .

Consider the case $\gamma c_1 - h > 0$ (which will be the case if salvage value is higher than holding cost and discount factor is close to 1). Then $\Psi(x, t)$ is monotonically non-decreasing in t . Note that $\Psi'_x(x_0, t) = -b$ if $t > x_0$ and $\Psi'_x(x_0, t) = h$ if $t < x_0$. Now let

$$t_0 := \inf \{t : \Psi(x_0, t) \geq \mathbb{E}[\Psi(x_0, D)]\}.$$

If $t_0 \geq x_0$ then (29) reduces to:

$$= \int_{t \geq t_0} \Delta(x_0, t) [c - b] dF(t) \leq 0, \tag{30}$$

where the inequality follows since $\Delta(x_0, t) \geq 0$ and $b > c$ for the problem to be nontrivial.

On the other hand, if $t_0 < x_0$ we can use the following inequality

$$- \int_{t \geq t_0} [c - b] dF(t) \geq \int_{x_0 \geq t \geq t_0} [c + h] dF(t), \tag{31}$$

which follows from the optimality conditions and the fact that $\Delta(x_0, t) \geq 0$ is non-decreasing in t , to conclude that $v'(x_0) \leq 0$. Therefore the minimizer x_λ of the newsvendor problem with a mean p -th semideviation objective is monotonically nondecreasing with λ (note that by an arbitrary small change of the cdf F we can ensure that the corresponding second order derivative $f''_0(x_0)$ exists and is nonzero). We have thus established the following result.

Corollary 2 *If $\gamma c_1 - h > 0$, then the minimizer x_λ of the newsvendor problem with mean p -th semideviation objective $\rho_\lambda[Z] := \mathbb{E}[Z] + \lambda (\mathbb{E}[Z - \mathbb{E}[Z]]_+^p)^{\frac{1}{p}}$ (with $p \geq 1$) is monotonically nondecreasing with λ .*

Let us check the sign of $v'(x_0)$ corresponding to $\mathbb{D}[Z] := CVaR_\alpha[Z]$. We assume again that the reference cdf $F(\cdot)$ is continuous. Let t_0 be the minimizer (assumed to be unique) of the right hand side of (11) corresponding to $Z = \Psi(x_0, D)$. Then

$$\frac{d}{dx} CVaR_\alpha[\Psi(x_0, D)] = \frac{d}{dx} \left(t_0 + (1 - \alpha)^{-1} \mathbb{E}_F[\Psi(x_0, D) - t_0]_+ \right). \quad (32)$$

Note that the variable t in the above formula can be fixed to the constant value t_0 by the so-called Danskin Theorem since the minimizer t_0 is assumed to be unique. It follows that we need to check the sign of

$$\frac{d}{dx} \mathbb{E}[\Psi(x_0, D) - t_0]_+ = \int_{\Psi(x_0, \tau) \geq t_0} \Psi'_x(x_0, \tau) dF(\tau). \quad (33)$$

Assuming that $\gamma c_1 - h > 0$, then $\Psi(x, \tau)$ is monotonically nondecreasing in τ . It is possible to use an argument similar to the one used to obtain (31) and conclude that $v'(x_0) \leq 0$. Then we have the following result.

Corollary 3 *If $\gamma c_1 - h > 0$, then the minimizer x_λ of the newsvendor problem with a mean-CVaR objective $\rho_\lambda[Z] := \mathbb{E}[Z] + \lambda CVaR_\alpha[Z]$ is monotonically nondecreasing with λ .*

If $\gamma c_1 - h < 0$ then $\Psi(x, t)$ is no longer guaranteed to be monotonic in t , and the sign of $v'(x_0)$ may be positive or negative. Intuitively, Corollaries 2 and 3 show that if the discounted salvage value is higher than the holding cost then increased risk aversion implies higher order quantity. Similar monotonicity results for the profit maximizing newsvendor model has been discussed in [2, 31].

3.5 Numerical illustration

We now present some numerical results for the newsvendor problem with the mean-absolute deviation objective (10). The problem parameters are as follows: ordering cost $c = 100$, holding cost $h = 20$, backordering cost $b = 60$ and discount factor $\gamma = 0.9$. The demand D is distributed according to a lognormal distribution with mean $\mu = 50$ and standard deviation $\sigma = 90$. Table 1 presents the optimal order quantity x^* for five different values of the

mean risk trade-off λ . Note that here $\gamma c_1 > h$, hence, as per Corollary 2, the optimal order quantity is increasing with λ .

Table 1: Optimal order quantity

λ	0	0.1	0.2	0.3	0.4
x^*	52.591	53.225	53.694	54.130	54.549

Next we compare the cost distribution of the risk neutral solution $x^* = 52.591$ (for $\lambda = 0$) and that of the risk averse solution $x^* = 54.549$ (for $\lambda = 0.4$) over a sample of 5000 demand scenarios generated for lognormal with $\mu = 50$ and $\sigma = 90$. To test the effect of imprecision in the underlying distribution, we also considered demand scenarios by changing μ and σ . Figure 2 presents the mean and range of the cost distribution for the two solutions corresponding to $\mu = 45, 50, 55$ and $\sigma = 90$. An immediate observation is the proportionality between the cost and the mean value. Clearly a bigger mean implies bigger costs. One can see that the difference between the average values for risk neutral and risk averse solutions is larger when the actual mean is less than the predicted mean. On the other hand if the actual mean is underestimated, then risk averse solution yields a lower average cost. In all cases the risk averse solution gives costs with smaller dispersion. As a result the risk averse solution dominates the risk neutral solution when the actual mean is underestimated. Figure 3 presents the mean and range of the cost distribution for the two solutions corresponding to $\mu = 50$ and $\sigma = 60, 90, 120$. Figure 3 shows the maximum, minimum and average cost for 5000 different scenarios generated using three different standard deviation values ($\sigma = 60, 90, 120$) and a fixed mean ($\mu = 50$). Note that the average value is not significantly affected by the change in standard deviation however the maximum value increases and the minimum value decreases as we increase the standard deviation. In all cases we end up with a smaller dispersion if we use the risk averse solution. These results demonstrate that the risk averse solution will yield costs with smaller dispersion even if the forecasted mean and standard deviation of the distribution are not accurate.

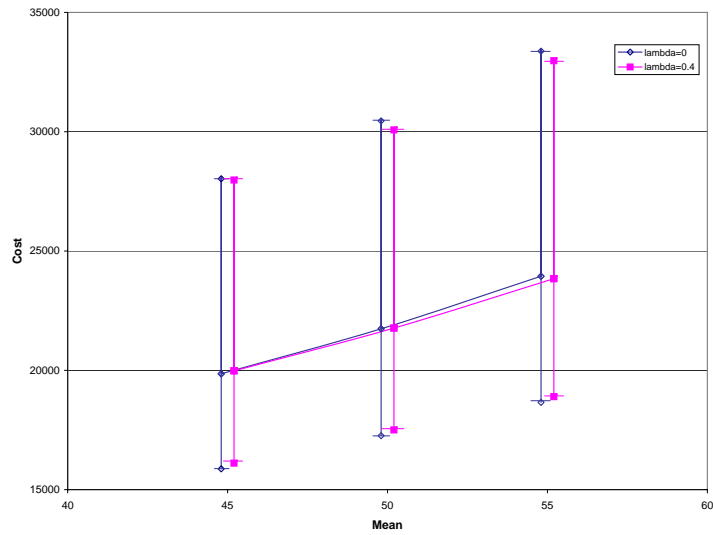


Figure 2: Maximum, minimum and average cost for different mean values

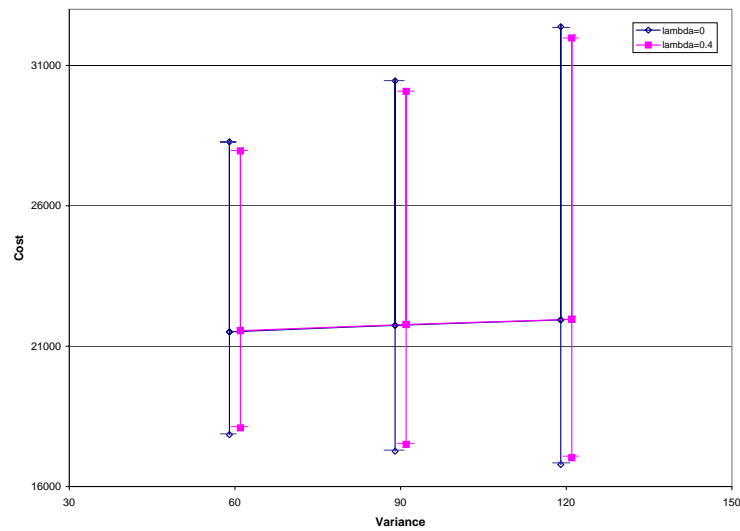


Figure 3: Maximum, minimum and average cost for different standard deviation values

CHAPTER IV

COHERENT RISK MAPPINGS IN MULTI-PERIOD INVENTORY PROBLEMS

4.1 Introduction

In this chapter we extend our work in Chapter 3 to a finite horizon multi-period setting. We first formulate the multi-period newsvendor problem in Section 4.2. Then we establish a one-to-one correspondence between the risk averse formulations using coherent risk measures and nested min-max type formulations. In Section 4.3 we analyze the optimal solution for the problem and prove the optimality of a base-stock policy. Hence as in single period case the optimal solution structure is same as that of the solution of the classical multi-period inventory problem. To obtain the optimal base-stock levels one needs to solve the dynamic programming equations. This is not an easy task in general. However, we provide conditions for the optimality of a myopic policy, in which case the optimal base-stock level can be obtained by solving a single stage problem. A more general version of the multi-period problem involving fixed ordering cost is also considered in Section 4.4. We show that even for this generalized version, our results on the optimal solution structure carry over.

4.2 Multi-period newsvendor models

Consider a planning horizon of T periods. In each period $t \in \{0, \dots, T\}$, the decision maker first observes the inventory level y_t and then places an order to replenish the inventory level to x_t ($\geq y_t$), i.e., the order quantity is $x_t - y_t$. The ordering cost is $c_t \geq 0$ per unit. After the inventory is replenished, demand d_t is realized and, accordingly, either (if $d_t < x_t$) an inventory holding cost of h_t per unit or (if $d_t \geq x_t$) a backorder penalty cost of b_t per unit is

incurred. The inventory holding and backorder penalty cost will be denoted by the function

$$\Psi_t(x_t, d_t) = b_t[d_t - x_t]_+ + h_t[x_t - d_t]_+. \quad (34)$$

Thus the total cost incurred in period t is $c_t(x_t - y_t) + \Psi_t(x_t, d_t)$. After demand is satisfied, the inventory level at the end of period t , i.e., at the beginning of period $t+1$ is $y_{t+1} = x_t - d_t$. It will be assumed that $b_t + h_t \geq 0$, $t = 0, \dots, T$, and hence functions $\Psi_t(x_t, d_t)$ are convex in x_t for any d_t .

We view the demand, considered as a function of time (period) t , as a random process D_t (as in the previous section we denote by d_t a particular realization of D_t). Unless stated otherwise we assume that the random process D_t is *across periods independent*, i.e., D_t is (stochastically) independent of (D_0, \dots, D_{t-1}) for $t = 1, \dots, T$. This assumption of across periods independence considerably simplifies the analysis. The cost of period t is discounted by a factor of γ^t where $\gamma \in (0, 1]$ is a given parameter. The remaining inventory y_{T+1} at the end of the planning horizon incurs a (discounted) cost of $-\gamma^{T+1}c_{T+1}y_{T+1}$.

In the classical risk neutral setting, the goal is to find an ordering policy to minimize expected total discounted cost. We consider a generalization of this classical model, where the expectation operation is replaced by a coherent risk measure ρ . Let us start our discussion with one period model. In the risk neutral setting the corresponding optimization problem can be formulated in the following form (compare with problem (16)):

$$\text{Min}_{x_0 \in \mathbb{R}} c_0(x_0 - y_0) + \mathbb{E}[\Psi_0(x_0, D_0) - \gamma c_1(x_0 - D_0)] \quad \text{subject to } x_0 \geq y_0, \quad (35)$$

where $y_0 \geq 0$ is a given initial value and the expectation is taken with respect to the probability distribution of D_0 . Note that by linearity of the expectation functional, the second term of the objective function of (35) can be written in the following equivalent form

$$\mathbb{E}[\Psi_0(x_0, D_0) - \gamma c_1(x_0 - D_0)] = \mathbb{E}[\Psi_0(x_0, D_0)] - \gamma c_1(x_0 - \mathbb{E}[D_0]).$$

Now for a specified coherent risk measure $\rho_0(\cdot)$ we can formulate the following risk-averse

analogue of problem (35):

$$\text{Min}_{x_0 \in \mathbb{R}} c_0(x_0 - y_0) + \rho_0 [\Psi_0(x_0, D_0) - \gamma c_1(x_0 - D_0)] \quad \text{subject to } x_0 \geq y_0. \quad (36)$$

Note that since $\rho_0(Z + a) = \rho_0(Z) + a$ for any constant $a \in \mathbb{R}$, we can write

$$\rho_0 [\Psi_0(x_0, D_0) - \gamma c_1(x_0 - D_0)] = \rho_0 [\Psi_0(x_0, D_0) + \gamma c_1 D_0] - \gamma c_1 x_0.$$

We discuss now an extension of the static problem (36) to a dynamic (multistage) setting. Let ρ_t , $t = 0, \dots, T$, be a sequence of coherent risk measures. We assume that risk measures ρ_t are *distribution invariant* in the sense that $\rho_t[Z]$ depends on the distribution of the random variable Z only. For example, we can use mean-absolute deviation risk measures

$$\rho_t(Z) := \mathbb{E}[Z] + \lambda_t \mathbb{E}[|Z_t - \mathbb{E}[Z_t]|], \quad (37)$$

where $\lambda_t \in [0, 1/2]$, $t = 0, \dots, T$, is a chosen sequence of numbers. Consider the following (nested) formulation of the corresponding multistage risk-averse problem:

$$\begin{aligned} \text{Min} \quad & c_0(x_0 - y_0) + \rho_0 [\Psi_0(x_0, D_0) + \gamma \rho_1 [c_1(x_1 - y_1) + \Psi_1(x_1, D_1) + \dots \\ & \gamma \rho_{T-1} [c_{T-1}(x_{T-1} - y_{T-1}) + \Psi_{T-1}(x_{T-1}, D_{T-1}) + \\ & \gamma \rho_T [c_T(x_T - y_T) + \Psi_T(x_T, D_T) - \gamma c_{T+1} y_{T+1}]]]] \\ \text{s.t.} \quad & x_t \geq y_t, \quad y_{t+1} = x_t - D_t, \quad t = 0, \dots, T. \end{aligned} \quad (38)$$

Note that axiom M3 in Definition 4 allows us to switch between this nested formulation and cumulative formulation. In fact the objective function in the model (38) is equivalent to

$$\tilde{\rho}(c_0(x_0 - y_0) + \Psi_0(x_0, D_0) + \dots + \gamma^T (c_T(x_T - y_T) + \Psi_T(x_T, D_T) - \gamma c_{T+1} y_{T+1})) \quad (39)$$

where $\tilde{\rho}$ is again a coherent risk measure.

Using the min-max representation (12) of ρ_t , $t = 0, \dots, T$, with \mathcal{A}_t being the corresponding set of cdf's, we can write (38) as

$$\begin{aligned}
\text{Min} \quad & c_0(x_0 - y_0) + \sup_{F \in \mathcal{A}_0} \mathbb{E}_F \left[\Psi_0(x_0, D_0) + \gamma \sup_{F \in \mathcal{A}_1} \mathbb{E}_F [c_1(x_1 - y_1) + \Psi_1(x_1, D_1) + \dots \right. \\
& \quad \left. \gamma \sup_{F \in \mathcal{A}_{T-1}} \mathbb{E}_F [c_{T-1}(x_{T-1} - y_{T-1}) + \Psi_{T-1}(x_{T-1}, D_{T-1}) + \right. \\
& \quad \left. \gamma \sup_{F \in \mathcal{A}_T} \mathbb{E}_F [c_T(x_T - y_T) + \Psi_T(x_T, D_T) - \gamma c_{T+1} y_{T+1}] \right] \\
\text{s.t.} \quad & x_t \geq y_t, \quad y_{t+1} = x_t - D_t, \quad t = 0, \dots, T.
\end{aligned} \tag{40}$$

We can write the corresponding dynamic programming equations for (38) as follows. At the last stage we need to solve the problem:

$$\text{Min}_{x_T \in \mathbb{R}} \rho_T [c_T(x_T - y_T) + \Psi_T(x_T, D_T) - \gamma c_{T+1}(x_T - D_T)] \quad \text{s.t.} \quad x_T \geq y_T. \tag{41}$$

Its optimal value, denoted $V_T(y_T)$, is a function of y_T . At stage $T - 1$ we solve the problem:

$$\begin{aligned}
\text{Min}_{x_{T-1} \in \mathbb{R}} \quad & \rho_{T-1} [c_{T-1}(x_{T-1} - y_{T-1}) + \Psi_{T-1}(x_{T-1}, D_{T-1}) + \gamma V_T(x_{T-1} - D_{T-1})] \\
\text{s.t.} \quad & x_{T-1} \geq y_{T-1}.
\end{aligned} \tag{42}$$

Its optimal value is denoted by $V_{T-1}(y_{T-1})$. And so on at stage $t = T - 1, \dots, 0$, we can write dynamic programming equations:

$$V_t(y_t) = \min_{x_t \geq y_t} \{c_t(x_t - y_t) + \rho_t [\Psi_t(x_t, D_t) + \gamma V_{t+1}(x_t - D_t)]\}. \tag{43}$$

Note that if each ρ_t is taken to be usual expectation operator (e.g., if we use ρ_t of the form (37) with all $\lambda_t = 0$), then the above becomes the standard formulation of a multistage inventory model (cf., [73]).

4.3 Optimal policy structure

A policy $x_t = x_t(d_0, \dots, d_{t-1})$, $t = 0, \dots, T$, is a function of a realization of the demand process up to time $t - 1$ (with $d_{-1} := 0$). Recalling that $y_t = x_{t-1} - d_{t-1}$, we can view a policy $x_t = x_t(y_t)$ as a function of y_t , $t = 0, \dots, T$. A policy is feasible if it satisfies the corresponding constraints $x_t \geq y_t$, $t = 0, \dots, T$. Because of the across periods independence

of the demand process, we have that, for a chosen policy $\hat{x}_t, \hat{y}_t = \hat{x}_{t-1} - D_{t-1}$, the total cost is given here by

$$\begin{aligned} & c_0(\hat{x}_0 - y_0) + \rho_0[\Psi_0(\hat{x}_0, D_0) + \gamma c_1(\hat{x}_1 - \hat{y}_1)] + \dots \\ & \rho_{T-1}[\Psi_{T-1}(\hat{x}_{T-1}, D_{T-1}) + \gamma c_T(\hat{x}_T - \hat{y}_T)] + \rho_T[\Psi_T(x_T, D_T) - \gamma c_{T+1}\hat{y}_{T+1}]. \end{aligned} \quad (44)$$

That is, the nested problem (38) can be formulated as minimization of the cost function (44) over all feasible policies.

By the dynamic programming equations (43) we have that a policy $\bar{x}_t = \bar{x}_t(y_t)$ is optimal iff

$$\bar{x}_t \in \arg \min_{x_t \geq y_t} \Lambda_t(x_t), \quad (45)$$

where

$$\Lambda_t(x_t) := \begin{cases} c_t x_t + \rho_t [\Psi_t(x_t, D_t) + \gamma V_{t+1}(x_t - D_t)], & t = 0, \dots, T-1, \\ (c_T - \gamma c_{T+1})x_T + \rho_T [\Psi_T(x_T, D_T) + \gamma c_{T+1}D_T], & t = T. \end{cases} \quad (46)$$

Theorem 4 For $t = 0, \dots, T$, let $x_t^* \in \arg \min_{x_t \in \mathbb{R}} \Lambda_t(x_t)$ be an unconstrained minimizer of $\Lambda_t(\cdot)$. Then the base-stock policy $\bar{x}_t := \max\{y_t, x_t^*\}$ solves the dynamic programming equations (44), and hence is optimal.

Proof. Since functions $\Psi_t(x_t, d_t)$ are convex in x_t , for any d_t , and ρ_t are convex and non-decreasing, it is straightforward to show by the induction that the value functions $V_t(\cdot)$ are convex, and hence functions $\Lambda_t(\cdot)$ are convex as well for all $t = 0, \dots, T$. By convexity of $\Lambda_t(\cdot)$ we have that if an unconstrained minimizer of $\Lambda_t(\cdot)$ is bigger than y_t , then it solves the right hand side of (44), otherwise $\bar{x}_t = y_t$ solves (44). \square

The result of the above theorem is based on convexity properties and does not require the assumption of across periods independence. It is also possible to write dynamic programming type equations for a general (not necessarily across periods independent) process (cf., [61]). In such a case the corresponding value functions $V_t(y_t, d_0, \dots, d_{t-1})$ will involve

a history of the demand process. Again, optimality of the corresponding base-stock policy will follow by convexity arguments.

Moreover, observe that in our proof we did not assume anything about the holding and backordering cost functions besides convexity. Although we utilized linear cost functions while building our model, in order for Theorem 4 to hold we only need convexity of functions $\Psi_t(x_t, d_t)$ in x_t for any d_t .

Theorem 5 *Suppose that the costs $c_t = c$, $t = 0, \dots, T + 1$, and parameters $b_t = b$, $h_t = h$, $t = 0, \dots, T$, are constant, and hence $\Psi_t(\cdot, \cdot) = \Psi(\cdot, \cdot)$ does not depend on t , that risk measures $\rho_t = \rho$, $t = 0, \dots, T$, are the same and that the demand process D_0, \dots, D_T is iid (independent identically distributed). Let*

$$x^* \in \arg \min_{x \in \mathbb{R}} \{(1 - \gamma)cx + \rho[\Psi(x, D) + \gamma cD]\}. \quad (47)$$

Then the myopic base-stock policy $\bar{x}_t := \max\{y_t, x^\}$ solves the dynamic programming equations (45), and hence is optimal.*

Proof. We have that x^* is an unconstrained minimizer of $\Lambda_T(\cdot)$, and hence by Theorem 4 the claim is true for $t = T$. We use now backward induction by t . Suppose the claim is true for some period t . Then by Theorem 4, $\Lambda_t(x^*) \leq \Lambda_t(x_t)$ for any x_t . Now, consider the period $t - 1$. By Theorem 4, the optimal policy is $\bar{x}_{t-1} = \max\{x_{t-1}^*, y_{t-1}\}$ where x_{t-1}^* is an unconstrained minimizer of

$$\begin{aligned} \Lambda_{t-1}(x_{t-1}) &= cx_{t-1} + \rho[\Psi(x_{t-1}, D) + \gamma V_t(x_{t-1} - D)] \\ &= (1 - \gamma c)x_{t-1} + \rho[\Psi(x_{t-1}, D) + \gamma cD + \gamma \Lambda_t(\max\{x^*, x_{t-1} - D\})], \end{aligned} \quad (48)$$

where the second line follows from the induction hypothesis and the translation equivariance property of ρ . Since the demand D is nonnegative, we clearly have that $\max\{x^*, x^* -$

$D\} = x^*$ for a.e. D . Then by (48) we have

$$\begin{aligned}
\Lambda_{t-1}(x^*) &= (1 - \gamma c)x^* + \rho[\Psi(x^*, D) + \gamma cD + \gamma \Lambda_t(\max\{x^*, x^* - D\})] \\
&= (1 - \gamma c)x^* + \rho[\Psi(x^*, D) + \gamma cD] + \gamma \Lambda_t(x^*) \\
&\leq (1 - \gamma c)x_{t-1} + \rho[\Psi(x_{t-1}, D) + \gamma cD] + \gamma \Lambda_t(x^*) \\
&\leq (1 - \gamma c)x_{t-1} + \rho[\Psi(x_{t-1}, D) + \gamma cD] + \gamma \Lambda_t(\max\{x^*, x_{t-1} - D\}) \text{ for a.e. } D,
\end{aligned} \tag{49}$$

where the second line follows from the translation equivariance property of ρ ; the third line from the definition of x^* ; and the fourth line follows from the induction hypothesis.

Let us observe now that if Z is a random variable such that $Z \geq \alpha$ w.p.1, then by the monotonicity property of ρ we have that $\rho[Z_1 + \alpha] \leq \rho[Z_1 + Z]$. Applying this with $Z := \Lambda_t(\max\{x^*, x_{t-1} - D\})$ and $Z_1 := \Psi(x_{t-1}, D) + \gamma cD$, we obtain by the last line of (49) that

$$\Lambda_{t-1}(x^*) \leq (1 - \gamma c)x_{t-1} + \rho[\Psi(x_{t-1}, D) + \gamma cD + \gamma \Lambda_t(\max\{x^*, x_{t-1} - D\})] = \Lambda_{t-1}(x_{t-1}), \tag{50}$$

for any x_{t-1} , where the last equality in (50) holds by (48). Thus $x_{t-1}^* = x^*$ minimizes $\Lambda_{t-1}(x_{t-1})$, and hence the result follows by Theorem 4. \square

We obtain that under the assumptions of the above theorem one can apply monotonicity results of the previous section to the optimal (myopic) policy in a straightforward way.

4.4 The multi-period problem with setup cost

We now consider the case when the ordering cost includes a fixed cost of k in each period. Thus the total cost incurred in period t is $k\varphi(x_t - y_t) + c_t(x_t - y_t) + \Psi_t(x_t, d_t)$, where $\varphi(x) = 1$ if $x > 0$ and 0 otherwise. In this case the dynamic programming recursion takes the following form.

$$V_t(y_t) = -c_t y_t + \min \{k\varphi(x_t - y_t) + \Lambda_t(x_t) : x_t \geq y_t\} \text{ for } t = 0, \dots, T, \tag{51}$$

where $\Lambda_t(x_t)$ are defined in the same way as in (46), and a policy $\bar{x}_t = \bar{x}_t(y_t)$ is optimal iff

$$\bar{x}_t \in \arg \min_{x_t \geq y_t} \{k\varphi(x_t - y_t) + \Lambda_t(x_t)\}, \quad t = 0, \dots, T. \tag{52}$$

Theorem 6 *Let for all $t = 0, \dots, T$,*

$$x_t^* \in \arg \min_{x \in \mathbb{R}} \Lambda_t(x) \text{ and } r_t^* := \max\{x : \Lambda_t(x) = k + \Lambda_t(x_t^*)\}. \quad (53)$$

Then the following policy is optimal for the dynamic program (51):

$$\bar{x}_t(y_t) := \begin{cases} x_t^*, & \text{if } y_t \leq r_t^*, \\ y_t, & \text{otherwise.} \end{cases} \quad (54)$$

If $k = 0$ then the above policy is a base-stock policy with base-stock level x_t^* , and if $k > 0$ the above policy is a (s, S) policy with reorder point $s = r_t^*$ and replenishment level $S = x_t^*$.

Theorem 6 follows from classical results if we can verify that, for all t , the functions $V_t(y_t)$ and $\Lambda_t(x_t)$ are k -convex in y_t and x_t respectively (cf., [73, section 9.5.2]). We shall need the following result.

Lemma 2 *If $f(x, d)$ is k -convex in x for all d (with $k \geq 0$) and ρ is a coherent risk measure, then $g(x) = \rho[f(x, D)]$ is k -convex.*

Proof. By the definition of k -convexity we have that $f(x, d)$ is k -convex in x iff for all $u, v > 0$ the following inequality holds:

$$\left(1 + \frac{u}{v}\right)f(x, d) - \frac{u}{v}f(x - v, d) \leq f(x + u, d) + k.$$

By the monotonicity and translation equivariance property of ρ it follows

$$\rho\left[\left(1 + \frac{u}{v}\right)f(x, D) - \frac{u}{v}f(x - v, D)\right] \leq \rho[f(x + u, D)] + k. \quad (55)$$

By convexity and positive homogeneity of ρ we have

$$\rho\left[\left(1 + \frac{u}{v}\right)f(x, D) - \frac{u}{v}f(x - v, D)\right] \geq \left(1 + \frac{u}{v}\right)\rho[f(x, D)] - \frac{u}{v}\rho[f(x - v, D)].$$

Together with (55) this implies

$$\left(1 + \frac{u}{v}\right)\rho[f(x, d)] - \frac{u}{v}\rho[f(x - v, d)] \leq \rho[f(x + u, d)] + k.$$

Thus $g(x) = \rho[f(x, D)]$ is k -convex. \square

Proof of Theorem 6. We only need to verify that, for all t , the functions $V_t(y_t)$ and $\Lambda_t(x_t)$ are k -convex in y_t and x_t respectively. By the nondecreasing convexity property of ρ , $\Lambda_T(x_T)$ is convex in x_T , and hence $V_T(y_T)$ is k -convex in y_T . Now suppose $V_t(y_t)$ and $\Lambda_t(x_t)$ are k -convex in y_t and x_t , respectively. Then $V_t(x_t - d_t)$ is k -convex in x_t since k -convexity is not affected by parallel shifts. Invoking Lemma 2 we have that $\Lambda_{t-1}(x_{t-1})$ is k -convex in x_{t-1} . Consequently $V_{t-1}(y_{t-1})$ is k -convex in y_{t-1} . \square

Chapters 3 and 4 are based on the paper "Coherent risk measures in inventory problems" by S. Ahmed, U. Çakmak and A. Shapiro, which is published in European Journal of Operational Research.

CHAPTER V

ROBUST INVENTORY PROBLEMS: INDEPENDENT UNCERTAINTY SETS CASE

5.1 Introduction

Our results for multi-period inventory models with coherent risk mappings indicates an equivalence between dynamic programming equations and nested min-max type models. Nested min-max type models are different from classical robust model because the decisions are not assumed to be fixed at the beginning of the planning horizon. In this chapter, we develop dynamic robust models in the context of inventory management with a special focus on the case where the uncertainty sets are independent of each other. In Section 5.2, we start with a definition of the problem and formulate the dynamic programming equations. Section 5.3 is dedicated to the analysis of static models and the comparison between static and dynamic versions. The connection between the dynamic robust models and coherent risk mappings provides valuable information helping us determining the optimal solution structure. In Section 5.4, we prove the optimality of base-stock policy for dynamic robust models and show that when the overage/underage costs are linear the problem is computational tractable. We also provide a closed form solution for a specific case with mild assumptions on problem parameters. Finally, the analysis of dynamic robust models is extended to the case where there is a fixed ordering cost in Section 5.5. For this generalized version, we prove the optimality of a (s, S) type policy.

5.2 Dynamic robust inventory problems

Consider the simple single item inventory problem in a multi-period setting. At the beginning of each period $t = 1, \dots, T$, inventory manager observes the inventory level y_t and

orders u_t units incurring a cost of c_t per unit, adding up to a total cost of $c_t u_t$. We assume that the demand at each period is random with an unknown distribution and may take any value in a given uncertainty set. We denote a particular realization of demand at period t by d_t and the uncertainty set covering all possible realizations by \mathcal{D}_t for all $t = 1, \dots, T$. Throughout this chapter we will assume that the demand process is *across periods independent*, i.e. the uncertainty set for demand at period t is independent of uncertainty sets for demand in previous periods $1, 2, \dots, t - 1$. Once demand is realized, either a holding cost based on units left in inventory or a backordering cost based on unmet demand is incurred. We denote this overage/underage cost by $\Psi_t(x_t, d_t)$ for $t = 1, \dots, T$, where x_t is the inventory level after replenishment, which is equal to $y_t + u_t$. Our only assumption on $\Psi_t(x_t, d_t)$ is convexity in x_t for any d_t . A typical example for such a function can be obtained by considering linear holding and backordering costs. The objective of the inventory manager is to minimize the maximum total cost, where the maximum is taken over all possible demand realizations.

One possible way of formulating this problem is to use robust optimization framework. Classical robust optimization models assume that all decision variables are fixed at the beginning of the planning horizon. As a result, these models are of static nature. It is clear from the above problem definition that in reality inventory manager will have the opportunity to update the order quantity decision at the beginning of each period. Hence he will be able to observe the demand realizations until the decision time. Naturally, one should take into account that the ordering decision may depend on observed demand realizations.

The idea of adjustable robust programming introduced by Ben-Tal *et al.* [11] is based on allowing the decision maker to determine future decisions as a function of part of uncertain data. Although the technique is developed for linear programs, it is possible to extend the basic idea to our setting. In the context of single item multi-period inventory problem this would imply expressing the order quantity decision at period t as a function of demand realizations until period t . Consequently, applying adjustable robust programming approach

to this problem will lead to a formulation featuring $\mathbf{u} = (u_1, u_2(d_1), \dots, u_T(d_1, \dots, d_{T-1}))$ as decision variables. Such a formulation would be a relaxation of its static counterpart since it takes into account all the flexibility available to the decision maker.

When feasibility is not an issue, as in the inventory problem we consider, the essence of adjustable robust modeling approach can be captured using a dynamic programming formulation. Our study focuses on this type of modeling methodology. Consider the single item multi-period inventory problem described above and assume that for every period and any quantity of unmet demand, backordering cost is larger than the cost of purchasing the same quantity. Let

$$\bar{V}_T(y_T) = \min_{u_T \geq 0} \left(c_T u_T + \max_{d_T \in \mathcal{D}_T} \Psi_T(y_T + u_T, d_T) \right) \quad (56)$$

be the value function for the last period. For each $t = 1, \dots, T - 1$ define

$$\bar{V}_t(y_t) = \min_{u_t \geq 0} \left(c_t u_t + \max_{d_t \in \mathcal{D}_t} [\Psi_t(y_t + u_t, d_t) + \bar{V}_{t+1}(y_t + u_t - d_t)] \right) \quad (57)$$

as cost-to-go function for period t . Note that the initial inventory \bar{y}_1 is given. Clearly, in this formulation u_t depends on y_t which in turn depends on demand realizations until period t .

By changing the choice of decision variable in these dynamic programming equations, we obtain another version that is more suitable for our analysis. Using x_t instead of u_t leads to the following formulation; let

$$V_T(y_T) = \min_{x_T \geq y_T} \left(c_T(x_T - y_T) + \max_{d_T \in \mathcal{D}_T} \Psi_T(x_T, d_T) \right) \quad (58)$$

be the value function for the last period and define

$$V_t(y_t) = \min_{x_t \geq y_t} \left(c_t(x_t - y_t) + \max_{d_t \in \mathcal{D}_t} [\Psi_t(x_t, d_t) + V_{t+1}(x_t - d_t)] \right) \quad (59)$$

for each $t = 1, \dots, T - 1$. It is easy to see that dynamic model given in (56, 57) is equivalent to the dynamic model given in (58, 59). Accordingly, from now on we will only use dynamic equations (58, 59) which will be referred as dynamic robust model.

Next we review some static modeling approaches and compare them against dynamic robust model.

5.3 Static models

Classical robust optimization models assume that the ordering quantities (u_t for $t = 1, \dots, T$) are fixed at the beginning of the planning horizon. Under this assumption the decision maker will order u_t at period t no matter what happens before t , even though this directly affects the initial inventory level at the time of decision. This leads to a static model ignoring the flexibility available in many real life multi-period problems. One possible static formulation for the multi-period single item inventory problem is the following

$$\begin{aligned}
\min_{\mathbf{u} \geq \mathbf{0}} \quad & \max_{\mathbf{d}, \mathbf{y}} \quad \sum_{t=1}^T (c_t u_t + \Psi_t(y_t + u_t, d_t)) \\
\text{s.t.} \quad & y_{t+1} = y_t + u_t - d_t \quad t = 1, \dots, T-1 \\
& y_1 = \bar{y}_1 \\
& \mathbf{d} \in \mathcal{D},
\end{aligned} \tag{60}$$

where \mathbf{u} , \mathbf{y} and \mathbf{d} are vectors whose elements are u_t , y_t and d_t , respectively, \mathcal{D} is the uncertainty set covering all possible demand realizations for all t and overage/underage cost $\Psi_t(\cdot, d_t)$ is defined as in previous sections. We assume that for every period and any quantity of unmet demand, backordering cost is larger than the cost of purchasing the same quantity to ensure positive ordering quantities and that the initial inventory \bar{y}_1 is given. Note that \mathbf{y} is not an actual decision variable but a function of \mathbf{u} and \mathbf{d} . It is possible to eliminate \mathbf{y} completely by replacing y_t with $\bar{y}_1 + \sum_{i=1}^{t-1} (u_i - d_i)$ for every $t = 2, \dots, T$. After this manipulation our formulation becomes

$$\min_{\mathbf{u} \geq \mathbf{0}} \quad \max_{\mathbf{d} \in \mathcal{D}} \quad \sum_{t=1}^T \left(c_t u_t + \Psi_t(\bar{y}_1 + \sum_{i=1}^t u_i - \sum_{i=1}^{t-1} d_i, d_t) \right). \tag{61}$$

The static problem is non-convex even for simple choices of cost functions and demand model, since the inner problem is non-convex with respect to \mathbf{d} . These formulations can be solved using Benders' decomposition based techniques, however there is no guarantee for computational tractability. Bienstock and Özbay [18] show that when the uncertainty

set \mathcal{D} is the cross product of closed intervals and the overage/underage cost is defined by linear holding and backordering cost functions the problem can be solved efficiently. On the other hand, they also state that for another demand uncertainty model (not satisfying the independence assumption) even the adversarial problem (finding the demand realizations that maximize the cost for given ordering quantities \mathbf{u}) is NP-hard.

When the overage/underage cost is composed of linear functions, a technique utilized to overcome computational difficulties is to consider an approximation of the static model. Assume that overage/underage cost $\Psi_t(\cdot, d_t)$ is defined as

$$\Psi_t(\cdot, d_t) = b_t[d_t - \cdot]_+ + h_t[\cdot - d_t]_+. \quad (62)$$

and that $b_t \geq c_t$ holds for every period. Then a conservative approximation of the static model can be formulated as follows:

$$\begin{aligned} \min \quad & \sum_{t=1}^T (cu_t + p_t) \\ \text{s.t.} \quad & p_t \geq h \left(\bar{y}_1 + \sum_{i=1}^t (u_i - d_i) \right) \quad \forall t = 1, \dots, T \\ & p_t \geq -b \left(\bar{y}_1 + \sum_{i=1}^t (u_i - d_i) \right) \quad \forall t = 1, \dots, T \\ & u_t \geq 0 \quad \forall t = 1, \dots, T, \end{aligned} \quad (63)$$

where the first two sets of constraints should be satisfied for all $\mathbf{d} \in \mathcal{D}$. Bertsimas and Thiele [17] studied this formulation and demonstrated that it is computationally tractable not only for the case where the uncertainty sets are independent of each other but also when they follow a specific dependency structure, namely budget of uncertainty structure. This dependency structure will be discussed in the next chapter, for now we focus on the model itself. The conservative model (63) brings the computational advantages of linear programs, however it is different than the original static model (60) since it tries to minimize an upper bound for the total cost. Observe that in the original model the maximum total

cost corresponds to a scenario defined by a T dimensional demand realizations vector (one realization for each period). In contrast, the conservative model (63) treats the holding and backordering costs separately for every period, allowing each period's overage/underage cost to correspond to a different scenario. This is equivalent to relaxing the constraints of adversarial problem, hence creating a conservative approximation of the original model. Naturally, if the difference between the original objective function and the upper bound utilized in conservative model is small, the approximation technique will be effective. We will demonstrate that this is not always the case.

We have presented three different modeling approaches and discussed the computational tractability of the static ones. Before analyzing the computational aspect of the dynamic model in detail, we will provide some examples to compare these modeling approaches and to prove the value of the dynamic one. Note that in all of our examples we assume that the holding and backordering cost functions are linear.

Example 2 Consider a 2 period problem where \mathcal{D} is given as $\mathcal{D}_1 \times \mathcal{D}_2$ and $\mathcal{D}_1 = \mathcal{D}_2 = [\bar{d} - \delta, \bar{d} + \delta]$. Assume that the cost parameters c, b and h are same for both periods and that they satisfy the following conditions: $b \geq c \geq h$ and $b = 3h$. For this problem the optimal solution of the conservative model (63) is to order $\bar{d} + \frac{\delta}{2}$ at the beginning of both periods. This follows from the optimality results in [17]. The maximum possible cost for these ordering quantities (which is also the optimal value of the conservative model) is $2\bar{d}c + \delta c + \frac{9}{2}\delta h$. The optimal solution of the original static model (60) is to order $\bar{d} + \delta$ in the first period and $\bar{d} - \frac{\delta}{2}$ in the second period. Note that in this problem the worst case will always occur at one of the four extreme points of \mathcal{D} regardless of ordering quantities. Accordingly, it is possible to determine the piecewise linear cost function and obtain the optimal solution. The maximum cost for this solution is $2\bar{d}c + \frac{\delta}{2}c + \frac{9}{2}\delta h$. Hence the gap between the optimal value of conservative model and that of the original static model is $\frac{\delta}{2}c$. Observe that this gap can be increased arbitrarily by increasing either δ or c . Hence the solution of conservative model does not necessarily provide a good solution for the original problem. Moreover,

if we utilize dynamic robust formulation (58, 59) for the same problem, by Theorem 9 the optimal solution is a base-stock policy with optimal inventory levels of $(\bar{d} + \delta, \bar{d} + \frac{\delta}{2})$. The corresponding optimal value is $2\bar{d}c + \frac{3\delta}{2}c + \frac{3\delta}{2}h$. The dynamic model outperforms the static one by a difference of $(b - c)\delta$, this gap can also be increased arbitrarily. This implies that even for a two period problem the dynamic model can significantly outperform the static models.

Theoretically, the dynamic model will always outperform its static counterparts. Example 2 demonstrates that the difference between the maximum cost corresponding to the optimal value of either one of the static models (63, 60) and the optimal value of the dynamic model (58, 59) may be unbounded, even for a simple two period problem. One might argue that these theoretical results have no practical value since they are obtained by artificially inflating problem parameters for a pathological case. Our next Example is numerical and it provides evidence on how much better the dynamic model can be in practice.

Example 3 Consider a 10 period single item inventory problem. Suppose that $\mathcal{D}_t = [30, 70]$ for every $t = 1, \dots, 10$ and that $c = 10$, $b = 12$ and $h = 4$ are the cost parameters (same for every period). The optimal solution of the static model (60) is $\mathbf{u}^* = (70, 70, 70, 70, 70, 70, 37.5, 0, 0, 0)$. The optimal value corresponding to this solution is 11,175 (4,575 ordering, 6,600 holding and backordering cost). If we consider the dynamic robust model (58, 59) under the same setting and apply Theorem 9, the optimal solution is a base-stock policy with the following levels $\mathbf{x}^* = (70, 70, 70, 70, 70, 70, 70, 70, 70, 60)$. The optimal value corresponding to this solution is 7,020 (6,900 ordering and 120 holding and backordering cost). In this example dynamic robust model outperforms its static counterpart by about 37%.

Our experience with similar examples shows that as long as T is greater than a few periods the dynamic robust model (58, 59) provides significant advantage. This is also intuitive since longer timeline implies even more flexibility. Note that the static model

provides a set of ordering quantities as the optimal solution rather than a policy. These ordering quantities are valid for all demand trajectories. In other words static models lack the flexibility of updating the decisions for different demand trajectories. One method employed to increase the performance of the static models is to use them on a rolling horizon basis. This allows the model to adjust itself according to the demand realizations. If we use the static model on a rolling horizon basis for the problem in Example 3, the optimal solution is a replicate of the optimal solution of dynamic robust model. The following example illustrates that rolling horizon approach does not eliminate all issues for static model.

Example 4 *Consider a 2 period single item inventory problem. Suppose that the demand uncertainty sets are given as $\mathcal{D}_1 = [30, 70]$ and $\mathcal{D}_2 = [10, 30]$ and that ordering costs are $c_1 = 5$ and $c_2 = 10$. Moreover, assume that $b = 12$ and $h = 4$ are the cost parameters (same for both periods) for overage/underage function. If we utilize the static model (60) on a rolling horizon basis the optimal solution for the first period will be $\mathbf{u}_1^* = 75$. Under the worst case scenario the total cost for this method will be 695. On the other hand the optimal solution for the dynamic robust model for the same problem is a base-stock policy with $x_1^* = 72$ and $x_2^* = 25$. This policy leads to a total cost of at most 658. This is an improvement of more than 5% over the static model applied on a rolling horizon basis.*

Our examples demonstrate that dynamic robust model not only theoretically outperforms static formulations but also provides significant practical benefits. The dynamic robust model may lower inventory related costs considerably even compared to static models employed in a rolling horizon approach.

5.4 Optimal solution structure

Since dynamic robust model is equivalent to an adjustable robust model applied to a non-convex problem, their computational complexity is similar. Unfortunately, the added flexibility of adjustable robust models has a price; increased complexity. While the classical

robust models are in general computationally tractable, only a small and simple class of adjustable robust models are guaranteed to possess this quality [11].

Most of the work on adjustable robust models considers problems that can be modeled as a linear program. Consequently, the complexity results are derived for adjustable robust linear programs. Note that the problem we consider is a non-convex problem that cannot be modeled as an adjustable robust linear program even if we assume linear overage and underage cost functions. Moreover, it is not easy to analyze the dynamic programming formulation for a general overage and underage cost function. Nevertheless, it is possible to obtain the structure of the optimal solution.

Consider the dynamic formulation given in (58, 59). Define

$$\Lambda_T(x_T) = c_T x_T + \max_{d_T \in \mathcal{D}_T} \Psi_T(x_T, d_T) \quad (64)$$

and

$$\Lambda_t(x_t) = c_t x_t + \max_{d_t \in \mathcal{D}_t} [\Psi_t(x_t, d_t) + V_{t+1}(x_t - d_t)] \quad t = 1, \dots, T-1, \quad (65)$$

where $V_t(\cdot)$ is as defined in (59).

Theorem 7 *Assume that \mathcal{D}_t is convex and let $x_t^* \in \operatorname{argmin}_{x_t \in \mathbb{R}} \Lambda_t(x_t)$ be an unconstrained minimizer of $\Lambda_t(x_t)$ for $t = 1, \dots, T$. Then the base-stock policy $\bar{x}_t := \max\{y_t, x_t^*\}$ solves the dynamic programming equations (58, 59), and hence is optimal.*

Proof. To prove this assertion we will first show the equivalence of dynamic programming equations (58, 59) to another set of dynamic programming equations involving coherent risk measures ρ_t instead of max operators. Recall the dual representation given in (12) and that Dirac measure δ_a is the measure of mass one at the point a . We will use these to eliminate max operator. We have for any given x_T

$$\max_{d_T \in \mathcal{D}_T} \Psi_T(x_T, d_T) = \max_{\mu \in \mathcal{A}_T} \mathbb{E}_\mu[\Psi_T(x_T, d_T)] = \rho_T(\Psi_T(x_T, d_T)) \quad (66)$$

where \mathcal{A}_T is defined as $\mathcal{A}_T := \operatorname{conv}\{\delta_a : a \in \mathcal{D}_T\}$. In this definition conv denotes the convex hull of the set of Dirac measures. The second equality in (66) follows by (12) because \mathcal{A}_T

is convex by definition. To see the validity of the first equality let d_T^* be the optimal solution of the problem $\max_{d_T \in \mathcal{D}_T} \Psi_T(x_T, d_T)$ for any given x_T . Hence $\Psi_T(x_T, d_T^*) \geq \Psi_T(x_T, d)$ for any $d \in \mathcal{D}_T$. By definition of \mathcal{A}_T , $\mathbb{E}_\mu[\Psi_T(x_T, d_T)]$ will be equal to convex combinations of $\Psi_T(x_T, d^i)$ for some set of $d^i \in \mathcal{D}_T$. The assumption that D_T is convex guarantees that every d^i belongs to D_T . Clearly, the convex combination cannot be greater than the maximum. It follows that $\mu = \delta_{d_T^*}$ is an optimal solution to $\max_{\mu \in \mathcal{A}_T} \mathbb{E}_\mu[\Psi_T(x_T, d_T)]$ and as a result the first equality holds. Consequently (58) can be rewritten as

$$V_T(y_T) = \min_{x_T \geq y_T} (c_T(x_T - y_T) + \rho_T(\Psi_T(x_T, d_T))), \quad (67)$$

for some coherent risk measure ρ_T . Observe that in the above argument we did not use any property of the function inside the first max operator in (66). Hence it is possible to use the same procedure for each t . It follows that for any $t = 1, \dots, T - 1$ and x_t

$$\max_{d_t \in \mathcal{D}_t} (\Psi_t(x_t, d_t) + V_{t+1}(x_t - d_t)) = \max_{\mu \in \mathcal{A}_t} \mathbb{E}_\mu[\Psi_t(x_t, d_t) + V_{t+1}(x_t - d_t)] \quad (68)$$

$$= \rho_t(\Psi_t(x_t, d_t) + V_{t+1}(x_t - d_t)) \quad (69)$$

holds. In (68) $\mathcal{A}_t := \text{conv}\{\delta_a : a \in \mathcal{D}_t\}$ and in (69) ρ_t is some coherent risk measure. This allows us to rewrite (59) as

$$V_t(y_t) = \min_{x_t \geq y_t} (c_t(x_t - y_t) + \rho_t(\Psi_t(x_t, d_t) + V_{t+1}(x_t - d_t))), \quad (70)$$

for each $t = 1, \dots, T - 1$. Observe that the functions $\Psi_t(x_t, d_t)$ are convex in x_t , for any d_t and ρ_t are convex and nondecreasing by definition of coherent risk measures. Using these properties and induction it is possible to show that value functions V_t given in Equations (67, 70) are convex, as in Theorem 4. Since these are equivalent to our original value functions we conclude that the functions Λ_t are convex as well for all $t = 0, \dots, T$. The convexity of these functions implies that maximum of x_t^* and y_t is optimal for the dynamic programming equations (58, 59). \square

Notice that the optimal base-stock levels are independent of demand history. Hence these levels are optimal for any demand trajectory, not only for the worst case.

This result on optimal solution structure is used below to prove the computational tractability of dynamic robust model with linear holding and backordering cost functions.

Theorem 8 *Assume that $\Psi_t(x_t, d_t)$ is defined as $b_t[d_t - x_t]_+ + h_t[x_t - d_t]_+$ for $t = 1, \dots, T$, where b_t and h_t are the unit backordering and holding cost, respectively. Then the dynamic programming equations (58, 59) can be solved efficiently.*

Proof. Consider the value function for the last period, $V_T(y_T)$. Since base-stock policy is optimal, when $y_T < x_T^*$ holds, $V_T(y_T)$ decreases with a slope of c_T as y_T increases. When $y_T \geq x_T^*$, $V_T(y_T)$ increases with a slope of h_T as y_T increases. We conclude that $V_T(y_T)$ is piecewise linear in y_T with 2 pieces.

Assume that $V_{t+1}(y_{t+1})$ is piecewise linear in y_{t+1} with n pieces, where n is polynomial in T and coefficients defining the pieces are polynomial in original problem parameters. Consider the value function for period t ,

$$V_t(y_t) = \min_{x_t \geq y_t} \left(c_t(x_t - y_t) + \max_{d_t \in \mathcal{D}_t} [\Psi_t(x_t, d_t) + V_{t+1}(x_t - d_t)] \right). \quad (71)$$

Clearly, the unconstrained problem to determine x_t^* can be solved efficiently. Observe that since base-stock policy is optimal, when $y_t < x_t^*$ holds, $V_t(y_t)$ decreases with a slope of c_t as y_t increases. This provides the first piece of the function. When $y_t \geq x_t^*$, $V_t(y_t)$ is simply $\max_{d_t \in \mathcal{D}_t} [\Psi_t(y_t, d_t) + V_{t+1}(y_t - d_t)]$. By definition $\Psi_t(y_t, d_t)$ is piecewise linear in y_t with 2 pieces. By assumption $V_{t+1}(y_t - d_t)$ is piecewise linear in y_t with n pieces. The sum of these two terms will be piecewise linear in y_t with at most $n + 1$ pieces. Moreover all coefficients defining these pieces were obtained from the coefficients of $V_{t+1}(y_{t+1})$ and the pair (b_t, h_t) . We conclude that $V_t(y_t)$ is piecewise linear in y_t with at most $n + 2$ pieces whose coefficients are polynomial in original problem parameters.

By induction it follows that the optimal base stock levels for the dynamic robust model can be obtained by solving T piecewise linear problem with at most $2T$ pieces. Hence the dynamic programming equations (58, 59) are computationally tractable. \square

We have proven that the dynamic robust model can be solved efficiently when back-ordering and holding costs are linear. In fact under some mild assumptions on problem parameters, it is even possible to provide closed form solution for the optimal base stock levels.

Assume without loss of generality that uncertainty set \mathcal{D}_t is an interval of the form $[\bar{d}_t - \delta_t, \bar{d}_t + \delta_t]$ for every $t = 1, \dots, T$, where \bar{d}_t is the nominal value of the demand and δ_t is the maximum deviation from the nominal value. Moreover, assume that the cost parameters satisfy $b_t \geq c_t \geq h_t$. Define the myopic optimal \tilde{x}_t for each $t = 1, \dots, T$ as follows

$$\tilde{x}_t = \operatorname{argmin}_{x_t \in \mathbb{R}} (c_t x_t + \max_{d_t \in \mathcal{D}_t} \Psi_t(x_t, d_t)) = \bar{d}_t + \frac{b_t - h_t}{b_t + h_t} \delta_t. \quad (72)$$

Note that since d_t lies in \mathcal{D}_t , it should be clear that \tilde{x}_t should lie in the same set as well. Observe that for any given order quantity the worst case scenario occurs in either lowest or highest demand points. Since lowering or increasing the order quantity effects the overall cost in opposite directions for these extreme demand points, it is possible to conclude that for the myopic optimal solution the worst case holding and backordering costs should be equal. For the ordering quantity defined in equation (72) the holding cost for lowest demand point and backordering cost for highest demand point are equal to $\frac{2h_t b_t}{b_t + h_t} \delta_t$.

Theorem 9 *Assume that uncertainty sets are of the form $\mathcal{D}_t = [\bar{d}_t - \delta_t, \bar{d}_t + \delta_t]$ for every $t = 1, \dots, T$, cost parameters satisfy $b_t \geq c_t \geq h_t$ and that $\tilde{x}_{t+1} \geq 2\delta_t$ for $t = 1, \dots, T - 1$. Furthermore suppose that $c_{t+1} \leq c_t$ for $t = 1, \dots, T - 1$. Then the optimal base-stock levels for the dynamic robust model (58, 59) are as follows*

$$x_T^* = \tilde{x}_T \quad (73)$$

$$x_t^* = \tilde{x}_t + \frac{2\delta_t \min\{c_{t+1}, h_t\}}{b_t + h_t} \quad t = 1, \dots, T - 1. \quad (74)$$

Proof. The fact that $x_T^* = \tilde{x}_T$ should be clear by definition of \tilde{x}_T and Λ_T given in (64). Suppose that $x_{t+1}^* \geq \tilde{x}_{t+1}$ holds for some t . Notice that if $\bar{d}_t - \delta_t \leq x_t^* \leq \bar{d}_t + \delta_t$, y_{t+1} will be less than $2\delta_t$ for any choice of d_t . By our assumptions this implies that y_{t+1} will be less than x_{t+1}^* . Now assume that $\bar{d}_t - \delta_t > x_t^*$. Then there exists $\gamma > 0$ such that $\bar{d}_t - \delta_t > x_t^* + \gamma$. We want to compare $\Lambda_t(x_t^*)$ and $\Lambda_t(x_t^* + \gamma)$. Notice that for any choice of d_t , $x_t^* - d_t \leq 0 \leq x_{t+1}^*$ holds and we have

$$V_{t+1}(x_t^* - d_t) = -c_{t+1}(x_t^* - d_t) + \Lambda_{t+1}(x_{t+1}^*), \quad (75)$$

and also $\Psi_t(x_t^*, d_t) = b_t(d_t - x_t^*)$. Consequently

$$\Lambda_t(x_t^*) = (c_t - c_{t+1})x_t^* + \max_{d_t \in \mathcal{D}_t} [b_t(d_t - x_t^*) + c_{t+1}d_t] + \Lambda_{t+1}(x_{t+1}^*). \quad (76)$$

A similar analysis for $x_t^* + \gamma$ shows that

$$\Lambda_t(x_t^* + \gamma) = \Lambda_t(x_t^*) - (c_{t+1} + b_t - c_t)\gamma. \quad (77)$$

Then $b_t \geq c_t$ and $\gamma > 0$ imply that $\Lambda_t(x_t^*) > \Lambda_t(x_t^* + \gamma)$, which contradicts with the optimality of x_t^* . Hence x_t^* cannot be smaller than $\bar{d}_t - \delta_t$. Suppose this time that $x_t^* > \bar{d}_t + \delta_t$. Then there exists $\gamma > 0$ such that $x_t^* - \gamma > \bar{d}_t + \delta_t$. Notice that in $\Lambda_t(x_t)$ the maximum will occur always on an extreme point, and it suffices to consider only the lowest and highest demand choices. Suppose the maximum occurs at the lowest demand $\bar{d}_t - \delta_t$. If $x_t^* - \bar{d}_t + \delta_t > x_{t+1}^*$ then we can choose γ so that $x_t^* - \gamma - \bar{d}_t + \delta_t > x_{t+1}^*$ holds. By the optimality of base-stock policy we have $V_{t+1}(x_t^* - \bar{d}_t + \delta_t) \geq V_{t+1}(x_t^* - \gamma - \bar{d}_t + \delta_t)$. Then

$$\Lambda_t(x_t^*) = c_t x_t^* + h_t(x_t^* - \bar{d}_t + \delta_t) + V_{t+1}(x_t^* - \bar{d}_t + \delta_t), \quad (78)$$

and for $x_t^* - \gamma$ we obtain

$$\Lambda_t(x_t^* - \gamma) \leq \Lambda_t(x_t^*) - (c_t + h_t)\gamma. \quad (79)$$

If $x_t^* - \bar{d}_t + \delta_t \leq x_{t+1}^*$, then we can use the same method used to obtain (77), noting that this time $\Psi_t(x_t^*, d_t) = h_t(x_t^* - d_t)$. We have

$$\Lambda_t(x_t^* - \gamma) = \Lambda_t(x_t^*) - (c_t + h_t - c_{t+1})\gamma. \quad (80)$$

Now suppose the maximum occurs at the highest demand $\bar{d}_t + \delta_t$. If $x_t^* - \bar{d}_t - \delta_t > x_{t+1}^*$ then we can choose γ so that $x_t^* - \gamma - \bar{d}_t - \delta_t > x_{t+1}^*$ holds. In this case we obtain the same inequality as (79). Otherwise equality (80) holds. This shows that regardless of the demand value that gives the maximum $\Lambda_t(x_t^* - \gamma) \leq \Lambda_t(x_t^*)$ will hold. Hence $x_t^* > \bar{d}_t + \delta_t$ cannot be true. It follows that $\bar{d}_t - \delta_t \leq x_t^* \leq \bar{d}_t + \delta_t$.

As a result our decision x_t^* will only affect the cost of ordering, holding and backordering at period t and the cost of ordering at period $t + 1$. More specifically x_t^* will be the minimizer of

$$\Lambda_t(x_t) = (c_t - c_{t+1})x_t + \max_{d_t \in \mathcal{D}_t} [\Psi_t(x_t, d_t) + c_{t+1}d_t] + \Lambda_{t+1}(x_{t+1}^*). \quad (81)$$

Notice that $\Psi_t(\tilde{x}_t, \bar{d}_t - \delta_t) = \Psi_t(\tilde{x}_t, \bar{d}_t + \delta_t)$. Clearly for \tilde{x}_t and any x value less than this the maximum will occur for the highest demand value $\bar{d}_t + \delta_t$. Consider $\tilde{x}_t - \gamma \geq \bar{d}_t - \delta_t$ for $\gamma > 0$. We have

$$\Lambda_t(\tilde{x}_t - \gamma) = \Lambda_t(\tilde{x}_t) + (b_t - c_t + c_{t+1})\gamma. \quad (82)$$

This implies that decreasing the base-stock level will increase the cost. On the other hand if we consider increasing the base-stock level by $\gamma > 0$, as long as the base-stock level is less than $\bar{d}_t + \delta_t$, the cost associated with highest demand decreases by $(b_t + c_{t+1} - c_t)\gamma$ whereas the cost associated with lowest demand increases by $(c_t + h_t - c_{t+1})\gamma$. At \tilde{x}_t the difference between these two cost values is $2\delta_t c_{t+1}$. Hence we can increase the base-stock level until this difference is eliminated or the base-stock level is equal to $\bar{d}_t + \delta_t$. The difference will be eliminated when

$$(c_t + h_t - c_{t+1})\gamma = 2\delta_t c_{t+1} + (c_t - b_t - c_{t+1})\gamma \quad (83)$$

holds. This equality is satisfied by $\gamma = (2\delta_t c_{t+1})/(b_t + h_t)$. The difference between $\bar{d}_t + \delta_t$ and \tilde{x}_t is $(2\delta_t h_t)/(b_t + h_t)$. We conclude that the optimal inventory level is

$$x_t^* = \tilde{x}_t + \min\left\{\frac{2\delta_t c_{t+1}}{b_t + h_t}, \frac{2\delta_t h_t}{b_t + h_t}\right\} = \tilde{x}_t + \frac{2\delta_t \min\{c_{t+1}, h_t\}}{b_t + h_t}. \quad (84)$$

Notice that $x_t^* \geq \tilde{x}_t$ is satisfied. Since this inequality holds for T by induction it will hold for any t . Hence the optimal base-stock levels for the dynamic robust model are as given in (73, 74). \square

Note that our conditions on problem parameters serve as separability conditions. They guarantee that there is no incentive for purchasing to meet future demand and no possibility of ending up with more initial inventory than unconstrained minimizer. As a result one can separate the value function for future periods from the current period problem.

The form we assume for the uncertainty sets do not cause any loss of generality and our conditions on cost parameters are mostly standard, however the assumption $\tilde{x}_{t+1} \geq 2\delta_t$ deserves more discussion. We consider the following example to clarify the value and implications of this assumption.

Example 5 *Consider a 2 period single item inventory problem. Suppose that the demand uncertainty sets are given as $\mathcal{D}_1 = [15, 75]$ and $\mathcal{D}_2 = [30, 60]$ and that ordering costs are $c_1 = c_2 = 10$. Moreover, assume that $b = 12$ and $h = 4$ are the cost parameters (same for both periods) for linear overage/underage functions. Note that the myopic optimal for 2nd period is $\tilde{x}_2 = 52.5$ and $\delta_1 = 30$. As a result, Theorem 9 does not apply. It is possible to solve the problem utilizing piecewise linear functions and the optimal base-stock levels are $x_1^* = 70, x_2^* = 52.5$ with an optimal value of 1,365. This is same as the optimal solution provided by Theorem 9 in equations (73, 74), which demonstrates that the condition $\tilde{x}_{t+1} \geq 2\delta_t$ is not a necessary but a sufficient condition. On the other hand, if we modify \mathcal{D}_1 to be $[10, 110]$, the optimal solution becomes $x_1^* = 107.83, x_2^* = 52.5$ with an optimal value of 1,741. This is different then the solution from equations (73, 74), which would imply a worst case cost of 1,780, only about 2 percent higher than the actual optimal value. Finally, consider the case where $\mathcal{D}_1 = [10, 500]$. For these uncertainty sets the optimal solution is $x_1^* = 393.83, x_2^* = 52.5$ with an optimal value of 6,889. The worst case scenario cost for the solution in (73, 74) is 8,800, almost 30 percent higher*

than the actual optimal value. In fact, it is possible to increase this gap in absolute terms by increasing δ_1 even more. However, by design that can only happen if nominal demand \bar{d}_1 is increased as well.

Example 5 demonstrates that the solution provided in Theorem 9 may be optimal even if the condition $\tilde{x}_{t+1} \geq 2\delta_t$ does not hold. Moreover, the example also suggests that the solution would perform fairly well when the uncertainty in a given period is not significantly higher than subsequent period's nominal demand. In other words, the closed form solution from Theorem 9 can be utilized whenever the demand and the variation is stable over time. However, bursty demand models where some periods have significantly higher nominal demands and variations will not be suitable for application of these results.

5.5 Fixed ordering cost case

Consider the case where the ordering cost includes a fixed cost of k in each period. Accordingly the total cost incurred in period t is $c_t(x_t - y_t) + k\phi(x_t - y_t) + \Psi_t(x_t, d_t)$, where $\phi(z) = 1$ if $z > 0$ and 0 otherwise. We obtain the dynamic programming equations for this problem by letting

$$V_T(y_T) = \min_{x_T \geq y_T} \left(c_T(x_T - y_T) + k\phi(x_T - y_T) + \max_{d_T \in \mathcal{D}_T} \Psi_T(x_T, d_T) \right) \quad (85)$$

be the value function for the last period and defining

$$V_t(y_t) = \min_{x_t \geq y_t} \left(c_t(x_t - y_t) + k\phi(x_t - y_t) + \max_{d_t \in \mathcal{D}_t} [\Psi_t(x_t, d_t) + V_{t+1}(x_t - d_t)] \right) \quad (86)$$

for each $t = 1, \dots, T - 1$. Note that

$$V_t(y_t) = -c_t y_t + \min_{x_t \geq y_t} (k\phi(x_t - y_t) + \Lambda_t(x_t))$$

holds for every $t = 1, \dots, T$, where $\Lambda_t(x_t)$ are defined as in (64, 65). We analyze the optimal solution structure for the dynamic programming equations (85, 86) using the approach in Section 4.4.

Theorem 10 Assume that \mathcal{D}_t is convex for every period. For all $t = 1, \dots, T$, define

$$x_t^* \in \operatorname{argmin}_{x_t \in \mathbb{R}} \Lambda_t(x_t)$$

and

$$r_t^* := \max\{x : \Lambda_t(x) = k + \Lambda_t(x_t^*)\}.$$

Then the optimal solution of the dynamic programming equations (85, 86) is a (s, S) policy with reorder point $s = r_t^*$ and replenishment level $S = x_t^*$.

When $k = 0$, this result is equivalent to Theorem 7. To prove it for $k > 0$, we will need to rewrite an equivalent form of dynamic programming equations (85, 86) using the technique we employed in the proof of Theorem 7 as follows

$$V_T(y_T) = \min_{x_T \geq y_T} (c_T(x_T - y_T) + k\phi(x_T - y_T) + \rho_T(\Psi_T(x_T, d_T))) \quad (87)$$

and

$$V_t(y_t) = \min_{x_t \geq y_t} (c_t(x_t - y_t) + k\phi(x_t - y_t) + \rho_t(\Psi_t(x_t, d_t) + V_{t+1}(x_t - d_t))) \quad (88)$$

for each $t = 1, \dots, T - 1$, where ρ_t is some coherent risk measure. Applying Theorem 6 to this formulation provides the optimality of a (s, S) policy.

CHAPTER VI

ROBUST INVENTORY PROBLEMS: DEPENDENT UNCERTAINTY SETS CASE

6.1 Introduction

So far in our analysis of robust inventory problems we have assumed that the uncertainty set \mathcal{D}_t for the demand of each period is independent of other uncertainty sets. In this chapter, we will relax this assumption and consider the case where uncertainty sets depend on each other. Besides this our problem definition remains same as in Section 5.2. We formulate dynamic programming equations for the problem with dependent uncertainty sets and discuss complexity in Section 6.2. In Section 6.3, we extend our result on the optimality of base-stock policy to the generalized dynamic robust model. In later sections, we focus on a specific dependency structure, namely budget of uncertainty. This dependency approach is described and modeled in Section 6.4. We propose a heuristic solution approach to solve dynamic robust model when budget of uncertainty approach is utilized as dependency structure. This heuristic approach proposed in Section 6.5 is also suitable for more general parametric dependency structures under certain conditions. Section 6.6 is devoted to cover modeling and solution techniques that can be used as an alternative to our proposed solution. Finally, Section 6.7 includes some computational results comparing our heuristic solution against the alternatives discussed.

6.2 Dynamic robust model with dependent uncertainty sets

The min-max based adjustable robust optimization model for the dependent uncertainty case can be formulated again by using dynamic programming. To obtain the dynamic robust model for this case we will need a few more definitions.

Define $\mathcal{D}^t = \{(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_t) : \exists(\tilde{d}_1, \dots, \tilde{d}_t, d_{t+1}, \dots, d_T) \in \mathcal{D}\}$ for $t = 1, 2, \dots, T-1$. Basically \mathcal{D}^t is the set of all possible demand trajectories up to period t . For any $t = 1, \dots, T-1$ and for any given $\tilde{d}^t = (\tilde{d}_1, \dots, \tilde{d}_t) \in \mathcal{D}^t$ define $\mathcal{D}_{\tilde{d}^t}^{t+1} := \{\tilde{d}_{t+1} : \exists(\tilde{d}_1, \dots, \tilde{d}_t, \tilde{d}_{t+1}, d_{t+2}, \dots, d_T) \in \mathcal{D}\}$, the set of all possible demand values for period $t+1$ if the demand realization up to period t is given by \tilde{d}^t . Finally, define $\mathcal{D}_0 = \{\tilde{d}_1 : \exists(\tilde{d}_1, d_2, \dots, d_T) \in \mathcal{D}\}$.

We will use these definitions to build our dynamic formulation as follows. Let

$$V_T(y_T, \tilde{d}^{T-1}) = \min_{x_T \geq y_T} \left(c_T(x_T - y_T) + \max_{d_T \in \mathcal{D}_{\tilde{d}^{T-1}}} \Psi_T(x_T, d_T) \right) \quad (89)$$

be the value function for the last period and define

$$V_t(y_t, \tilde{d}^{t-1}) = \min_{x_t \geq y_t} \left(c_t(x_t - y_t) + \max_{d_t \in \mathcal{D}_{\tilde{d}^{t-1}}} [\Psi_t(x_t, d_t) + V_{t+1}(x_t - d_t, (\tilde{d}^{t-1}, d_t))] \right) \quad (90)$$

for each $t = 2, 3, \dots, T-1$ and the first period value function for given initial inventory y_1

$$V_1(y_1) = \min_{x_1 \geq y_1} \left(c_1(x_1 - y_1) + \max_{d_1 \in \mathcal{D}_0} [\Psi_1(x_1, d_1) + V_2(x_1 - d_1, d_1)] \right). \quad (91)$$

Note that this model is generalized both in terms of demand structure and overage/underage costs. As a result it is difficult to analyze the computational complexity. Even for simple choices of demand and cost structure the problem becomes non-convex. Moreover, the dynamic programming equations usually suffer from non-separability.

In fact even the results on adjustable robust linear models are not promising. Specifically, a work by Guslitzer [42] proves that an adjustable robust linear program is computationally tractable if the following conditions hold:

1. The uncertainty set is given as a convex hull of a finite number of scenarios.
2. The coefficients of adjustable variables are independent of uncertainty.

The same work [42] also demonstrates that in case either one of these conditions is not satisfied the adjustable robust linear program can be NP-Complete even for problems with very simple structures. In our formulation the first condition is not satisfied, hence even if

we had a linear model, under our problem setting computational tractability would not be expected in most cases. Nevertheless, we will see next that our results on optimal policy from chapter 5 can be generalized for dependent uncertainty sets case.

6.3 Optimality of base-stock policy

Theorem 11 *Consider the dynamic programming equations (89, 90, 91), assume that \mathcal{D}_0 and $\mathcal{D}_{\tilde{d}^t}$ is convex for any $t = 1, \dots, T-1$ and any $\tilde{d}^t \in \mathcal{D}^t$. Then the base-stock policy $\bar{x}_t := \max\{y_t, x_t^*\}$, where x_t^* is the unconstrained minimizer of equations (89, 90, 91) for $t = 1, \dots, T$, is optimal for the dynamic programming.*

Proof. We will use the same strategy used in the proof of Theorem 7. For any given x_T and \tilde{d}^{T-1} we have

$$\max_{d_T \in \mathcal{D}_{\tilde{d}^{T-1}}} \Psi_T(x_T, d_T) = \max_{\mu \in \mathcal{A}_{\tilde{d}^{T-1}}} \mathbb{E}_\mu[\Psi_T(x_T, d_T)] = \rho_{\tilde{d}^{T-1}}(\Psi_T(x_T, d_T)) \quad (92)$$

where $\mathcal{A}_{\tilde{d}^{T-1}}$ is defined as $\mathcal{A}_{\tilde{d}^{T-1}} := \text{conv}\{\delta_a : a \in \mathcal{D}_{\tilde{d}^{T-1}}\}$. Note that by our assumption $\mathcal{D}_{\tilde{d}^{T-1}}$ is convex for any \tilde{d}^{T-1} . The only difference between (92) and (66) is that this time the coherent risk measure that we obtain will depend on realizations of uncertain demand up to current period. But conditional mappings allow us to write down dynamic equations for this situation. Clearly, for any x_t and \tilde{d}^{t-1} we have the following

$$\begin{aligned} \max_{d_t \in \mathcal{D}_{\tilde{d}^{t-1}}} (\Psi_t(x_t, d_t) + V_{t+1}(x_t - d_t, (\tilde{d}^{t-1}, d_t))) &= \max_{\mu \in \mathcal{A}_{\tilde{d}^{t-1}}} \mathbb{E}_\mu[\Psi_t(x_t, d_t) + V_{t+1}(x_t - d_t, (\tilde{d}^{t-1}, d_t))] \\ &= \rho_{\tilde{d}^{t-1}}(\Psi_t(x_t, d_t) + V_{t+1}(x_t - d_t, (\tilde{d}^{t-1}, d_t))), \end{aligned}$$

and finally for the first period we have

$$\begin{aligned} \max_{d_1 \in \mathcal{D}_0} (\Psi_1(x_1, d_1) + V_2(x_1 - d_1, d_1)) &= \max_{\mu \in \mathcal{A}_0} \mathbb{E}_\mu[\Psi_1(x_1, d_1) + V_2(x_1 - d_1, d_1)] \\ &= \rho_0(\Psi_1(x_1, d_1) + V_2(x_1 - d_1, d_1)). \end{aligned}$$

Consequently, it is possible to rewrite the dynamic equations (89, 90, 91) using coherent

risk measures as follows

$$V_T(y_T, \tilde{d}^{T-1}) = \min_{x_T \geq y_T} (c_T(x_T - y_T) + \rho_{\tilde{d}^{T-1}}(\Psi_T(x_T, d_T))), \quad (93)$$

$$V_t(y_t, \tilde{d}^{t-1}) = \min_{x_t \geq y_t} (c_t(x_t - y_t) + \rho_{\tilde{d}^{t-1}}(\Psi_t(x_t, d_t) + V_{t+1}(x_t - d_t, (\tilde{d}^{t-1}, d_t)))) \quad (94)$$

$$V_1(y_1) = \min_{x_1 \geq y_1} (c_1(x_1 - y_1) + \rho_0(\Psi_1(x_1, d_1) + V_2(x_1 - d_1, d_1))), \quad (95)$$

where (94) is for $t = 2, \dots, T-1$. The overage/underage cost functions $\Psi_t(x_t, d_t)$ are convex in x_t for any d_t by our assumption. Moreover, ρ_0 and $\rho_{\tilde{d}^t}$ for $t = 1, \dots, T-1$ are convex and nondecreasing for any \tilde{d}^t by definition of coherent risk measures. Utilizing these results and induction we have that the value functions V_t given in Equations (93, 94, 95) are convex in y_t for any \tilde{d}^t . Hence the objective function of unconstrained minimization problems are convex in x_t and the optimality of base-stock policy follows. \square

Note that the optimal base-stock levels for the dependent uncertainty sets case are state dependent. In other words x_t^* depends on \tilde{d}^{t-1} for $t = 2, \dots, T$. This is an expected result since the problem that inventory manager faces at the beginning of period t depends on the demand history until that period, and the dependence is not limited to the initial inventory. The demand history shapes the uncertainty sets for the demand of future periods and inventory decisions will clearly depend on these sets.

6.4 Budget of uncertainty approach

It is hard to analyze the dynamic robust model for dependent uncertainty sets case for a general dependence relationship. In this section, we will focus on a specific dependency structure. Budget of uncertainty approach [17] is based on bounding the length of the intervals that demand values lie. In this approach we assume that $d_t \in \mathcal{D}'_{z_t} = [\bar{d}_t - \delta_t z_t, \bar{d}_t + \delta_t z_t]$, where \bar{d}_t is the nominal value for demand and δ_t is the maximum possible deviation from the nominal value for each $t = 1, \dots, T$. Notice that here we have an additional term z_t that does not exist in the independent case. This term is a decision variable that

determines the length of interval that d_t belongs to. An important criticism against robust models is their over-conservative nature. Specifically, in multi-period robust models there is an inherent assumption that the worst case scenario will occur in every period. Budget of uncertainty approach tries to address this issue by tying the uncertainty sets together using z_t as a factor limiting the combined deviation. Moreover, the linear relationship between demands across periods promises more in terms of computational tractability than more complex dependency structures.

Clearly, if we let z_t be independent from each other and assume that $z_t \leq 1$ for every t then we obtain independent uncertainty sets case. In budget of uncertainty approach we assign a budget level Γ_t to each period $t = 1, \dots, T$. These budget levels should satisfy $\Gamma_{t+1} \geq \Gamma_t$ and $\Gamma_{t+1} - \Gamma_t \leq 1$ for $t = 0, \dots, T-1$ (let $\Gamma_0 = 0$). Define the polytope \mathcal{Z}_j for $j = 1, \dots, T$ as follows

$$0 \leq z_t \leq 1 \quad \forall t = 1, \dots, j, \quad (96)$$

$$\sum_{t=1}^k z_t \leq \Gamma_k \quad \forall k = 1, \dots, j, \quad (97)$$

and assume that $\mathbf{z} = (z_1, \dots, z_T)$ is an element of \mathcal{Z}_T . For any $t = 1, \dots, T-1$ and for any given $\bar{z}^t = (\bar{z}_1, \dots, \bar{z}_t) \in \mathcal{Z}_t$ define $\mathcal{Z}_{\bar{z}^t} := \{\bar{z}_{t+1} : (\bar{z}_1, \dots, \bar{z}_t, \bar{z}_{t+1}) \in \mathcal{Z}_{t+1}\}$. This is the set of all possible choices of z_{t+1} for a given \bar{z}^t . Let $\mathcal{Z}_0 = \mathcal{Z}_1$. Now we can formulate the dynamic programming equations for this problem as follows. Let

$$V_T(y_T, \bar{z}^{T-1}) = \min_{x_T \geq y_T} \left(c_T(x_T - y_T) + \max_{z_T \in \mathcal{Z}_{\bar{z}^{T-1}}, d_T \in \mathcal{D}_{z_T}^T} \Psi_T(x_T, d_T) \right) \quad (98)$$

be the value function for the last period, define

$$V_t(y_t, \bar{z}^{t-1}) = \min_{x_t \geq y_t} \left(c_t(x_t - y_t) + \max_{z_t \in \mathcal{Z}_{\bar{z}^{t-1}}, d_t \in \mathcal{D}_{z_t}^t} [\Psi_t(x_t, d_t) + V_{t+1}(x_t - d_t, (\bar{z}^{t-1}, z_t)) \right] \quad (99)$$

for each $t = 2, 3, \dots, T-1$ and the first period value function for given initial inventory y_1

$$V_1(y_1) = \min_{x_1 \geq y_1} \left(c_1(x_1 - y_1) + \max_{z_1 \in \mathcal{Z}_0, d_1 \in \mathcal{D}_{z_1}^1} [\Psi_1(x_1, d_1) + V_2(x_1 - d_1, z_1)] \right). \quad (100)$$

Note that we do not need to use a separate decision variable d_t for the demand since with this formulation d_t will always be equal to either $\bar{d}_t - \delta_t z_t$ or $\bar{d}_t + \delta_t z_t$. We have chosen to formulate the problem as above so that we can easily modify the problem. The dynamic problem given in (98, 99, 100) is non-convex even if we assume linear holding and backordering cost functions. Efficiently solving the dynamic programming equations requires eliminating the non-separability issue. Instead we suggest a heuristic method.

6.5 Proposed heuristic solution

Assume linear holding and backordering cost functions. Observe that if we fix $\mathbf{z} = (z_1, \dots, z_T) \in \mathcal{Z}_T$ at the beginning, then we obtain independent uncertainty sets. We have seen in Section 5.4 that we can solve this problem and obtain closed form solutions under certain assumptions. For fixed z_t 's, let

$$V_T(y_T, z_T) = \min_{x_T \geq y_T} \left(c_T(x_T - y_T) + \max_{d_T \in \mathcal{D}_{z_T}} \Psi_T(x_T, d_T) \right) \quad (101)$$

be the value function for the last period and define

$$V_t(y_t, z_t, \dots, z_T) = \min_{x_t \geq y_t} \left(c_t(x_t - y_t) + \max_{d_t \in \mathcal{D}_{z_t}} [\Psi_t(x_t, d_t) + V_{t+1}(x_t - d_t, z_{t+1}, \dots, z_T)] \right) \quad (102)$$

for each $t = 1, \dots, T - 1$. The modified problem is

$$\max_{(z_1, \dots, z_T) \in \mathcal{Z}_T} V_1(y_1, z_1, \dots, z_T) \quad (103)$$

The formulation (103) is not equivalent to the original problem (98, 99, 100), it is a computationally tractable approximation. Here we will show how to obtain an optimal solution of this modified problem. We will test the quality of this solution and compare it against alternatives later in this chapter.

Theorem 12 *Assume that $\bar{d}_{t+1} \geq 2\delta_t$ for $t = 1, \dots, T - 1$. Furthermore suppose that $h_t \leq c_{t+1} \leq c_t \leq b_t$ for $t = 1, \dots, T - 1$. Then an optimal solution of problem (103) can be*

found by solving the following linear program

$$\begin{aligned} \max \quad & \sum_{t=1}^{T-1} (c_t \delta_t z_t) + \left(\frac{c_T(b_T - h_T) + 2b_T h_T}{b_T + h_T} \right) \delta_T z_T \\ \text{s.t.} \quad & (z_1, z_2, \dots, z_T) \in \mathcal{Z}_T. \end{aligned} \quad (104)$$

Proof. Consider any fixed z_1, \dots, z_T . Notice that in this case we have to change the definition of myopic optimal solution (72) as follows

$$\tilde{x}_t = \operatorname{argmin}_{x_t \in \mathbb{R}} (c_t x_t + \max_{d_t \in \mathcal{D}_{z_t}} \Psi_t(x_t, d_t)) = \bar{d}_t + \frac{b_t - h_t}{b_t + h_t} \delta_t z_t. \quad (105)$$

This implies that regardless of choice of z_t 's $\tilde{x}_t \geq \bar{d}_t \geq 2\delta_t \geq \delta_t z_t$ should hold. Then it is possible to use the solution given in Theorem 9. Note that additional assumption $h_t \leq c_{t+1}$ implies that we will have the following closed form solution for given z_t values

$$x_T^* = \bar{d}_T + \frac{b_T - h_T}{b_T + h_T} \delta_T z_T \quad (106)$$

$$x_t^* = \bar{d}_t + \delta_t z_t \quad t = 1, \dots, T-1. \quad (107)$$

From the proof of Theorem 9 one can also deduce that the worst case scenario for this solution occurs when the highest possible demand is chosen for each period. As a result it is possible to write down the value function explicitly as

$$V_1(y_1, z_1, \dots, z_T) = \sum_{t=1}^{T-1} \left(c_t (\bar{d}_t + \delta_t z_t) \right) + c_T \left(\bar{d}_T + \frac{b_T - h_T}{b_T + h_T} \delta_T z_T \right) + \frac{2b_T h_T}{b_T + h_T} \delta_T z_T. \quad (108)$$

To maximize this value one can use the linear programming formulation (104). \square

Note that our assumptions in Theorem 12 are mainly designed to make sure that the problem (103) satisfies all conditions listed in Theorem 9. The discussion on the assumption $\bar{d}_{t+1} \geq 2\delta_t$ at the end of the proof of Theorem 9 demonstrates that even if the assumption does not hold the solution provided might be optimal or provide small optimality gap as long as the demand model does not include any significant peak periods. Accordingly, as long as demand and variation are stable across periods, we would expect our heuristic to

provide optimal or near optimal solutions for problem (103) even if $\bar{d}_{t+1} \geq 2\delta_t$ does not hold.

The optimal solution \mathbf{z}^* for the modified problem will also give us an implementable base-stock policy with base-stock levels \mathbf{x}^* . These levels can be obtained by plugging the optimal z_t^* values into (106) and (107).

Theorem 13 *The value function $V_1(y_1, z_1, \dots, z_T)$ defined in (108) evaluated at \mathbf{z}^* gives a lower bound for the optimal value of the original problem $V_1(y_1)$ given in (100). As an upper bound one can use the optimal value of the following problem*

$$\begin{aligned} \max_{\mathbf{z}, \mathbf{d}, \mathbf{y}} \quad & \sum_{t=1}^T (c_t(x_t^* - y_t) + \Psi_t(x_t^*, d_t)) \\ \text{s.t.} \quad & y_{t+1} = x_t^* - d_t \quad t = 1, \dots, T-1 \\ & d_t \in \mathcal{D}_{z_t} \quad t = 1, \dots, T \\ & \mathbf{z} = (z_1, \dots, z_T) \in \mathcal{Z}_T, \end{aligned} \tag{109}$$

where \mathbf{x}^* is given by the solution of the modified problem.

Proof. Notice that \mathbf{x}^* represents a feasible static ordering policy and the optimal value of (109) is simply the maximum possible cost for this policy. Hence it is an upper bound for the optimal value of the static version of (98, 99, 100). Since dynamic version is a relaxation of static version it is also an upper bound for $V_1(y_1)$.

Now consider $V_1(y_1, z_1, \dots, z_T)$ evaluated at \mathbf{z}^* which gives the optimal value of the modified problem. Observe that modified problem can be obtained from the original problem (98, 99, 100) by interchanging min and max operators. In particular one should replace min max with max min iteratively. By duality the optimal value of the problem obtained in every iteration is smaller than the optimal value of the problem before iteration. Hence the optimal value of the modified problem is a lower bound for $V_1(y_1)$. \square

Although the proposed solution methodology is devised for budget of uncertainty approach, it can be applied to general parametric dependency structures. Consider a set of

parameters \mathcal{P} and assume that every element \mathbf{p} of \mathcal{P} defines T independent demand intervals $\mathcal{D}_{\mathbf{p}}^1, \mathcal{D}_{\mathbf{p}}^2, \dots, \mathcal{D}_{\mathbf{p}}^T$. Let $\mathcal{D}_{\mathbf{p}}$ be the cross product of these intervals and define $\mathcal{D} := \{\mathcal{D}_{\mathbf{p}} : \mathbf{p} \in \mathcal{P}\}$. The methodology outlined in this section can be applied to a dynamic robust model where the backordering and holding costs are linear and the uncertainty set \mathcal{D} is as defined above if the following conditions hold:

1. Let $u_{\mathcal{D}_{\mathbf{p}}^t}$ and $l_{\mathcal{D}_{\mathbf{p}}^t}$ be the highest and lowest possible demand values in the interval $\mathcal{D}_{\mathbf{p}}^t$. The inequality $(u_{\mathcal{D}_{\mathbf{p}}^{t+1}} + l_{\mathcal{D}_{\mathbf{p}}^{t+1}}) \geq (u_{\mathcal{D}_{\mathbf{p}}^t} - l_{\mathcal{D}_{\mathbf{p}}^t})$ should hold for every $\mathbf{p} \in \mathcal{P}$ and $t = 1, \dots, T - 1$.
2. For every $t = 1, \dots, T - 1$, the condition $h_t \leq c_{t+1} \leq c_t \leq b_t$ should be satisfied.
3. The modified problem

$$\max_{\mathbf{p} \in \mathcal{P}} V_1(y_1, \mathbf{p})$$

should be computationally tractable.

First two conditions allow us to use the closed form solution developed in Section 5.4. The third condition is a general one and holds for a wide variety of parametric dependency structures. For example if we consider the budget of uncertainty approach but assume that z_t 's are elements of a T dimensional convex set, the last condition will be satisfied.

6.6 *Alternative models and solutions*

There are a limited number of approaches one can utilize as an alternative to our proposed method. A natural choice is to use the static model (60) and define \mathcal{D} according to budget of uncertainty framework as follows:

$$(d_1, \dots, d_T) \in \mathcal{D}, \quad (110)$$

$$d_t \in [\bar{d}_t - \delta_t z_t, \bar{d}_t + \delta_t z_t] \quad \forall t = 1, \dots, T,$$

$$0 \leq z_t \leq 1 \quad \forall t = 1, \dots, T,$$

$$\sum_{t=1}^k z_t \leq \Gamma_k \quad \forall k = 1, \dots, T.$$

Although this model is not necessarily computationally tractable, it is possible to solve fairly large problem instances using Benders' decomposition based algorithms. Bienstock and Ozbay [18] conjecture that a broad classes of such problems can be solved efficiently. To improve the performance, the static model can be used on a rolling horizon basis.

Another approach is to use a conservative linear approximation of the static model to ensure computational tractability. Bertsimas and Thiele [17] considers the following formulation

$$\begin{aligned} \min \quad & \sum_{t=1}^T (cu_t + p_t) \\ \text{s.t.} \quad & p_t \geq h \left(\bar{y}_1 + \sum_{i=1}^t (u_i - d_i) \right) \quad \forall t = 1, \dots, T, \\ & p_t \geq -b \left(\bar{y}_1 + \sum_{i=1}^t (u_i - d_i) \right) \quad \forall t = 1, \dots, T, \\ & u_t \geq 0 \quad \forall t = 1, \dots, T, \end{aligned} \quad (111)$$

where the first two sets of constraints should be satisfied for all $\mathbf{d} \in \mathcal{D}$ and \mathcal{D} is defined in (110). This conservative model differs from the original static model in that it tries to minimize an upper bound for the total cost. While the conservative version (111) is still a static model, the non-convexity is eliminated, allowing a computationally tractable model. In fact, using strong duality this problem can be converted to the following linear program:

$$\begin{aligned}
\min \quad & \sum_{t=1}^T (cu_t + p_t) \\
\text{s.t.} \quad & p_t \geq h \left(\bar{y}_1 + \sum_{i=1}^t (u_i - \bar{d}_i) + q_t \Gamma_t + \sum_{i=1}^t r_{it} \right) \quad \forall t = 1, \dots, T, \\
& p_t \geq -b \left(\bar{y}_1 + \sum_{i=1}^t (u_i - \bar{d}_i) - q_t \Gamma_t - \sum_{i=1}^t r_{it} \right) \quad \forall t = 1, \dots, T, \\
& q_t + r_{it} \geq \delta_i \quad \forall t = 1, \dots, T, \quad \forall i \leq t, \\
& q_t \geq 0, \quad r_{it} \geq 0 \quad \forall t = 1, \dots, T, \quad \forall i \leq t, \\
& u_t \geq 0 \quad \forall t = 1, \dots, T.
\end{aligned} \tag{112}$$

To decrease the effect of the static nature of the model, rolling horizon approach can be employed with this conservative model as well.

Both alternative methods discussed above are mainly of static nature. A method based on approximating the dynamic version of the problem uses affinely adjustable robust models. These models are introduced by Ben-Tal *et al.* [11] as a computationally tractable alternative to adjustable robust models. In an adjustable robust model there is no specific structure on how the future decisions are related to the random factors. The idea behind affinely adjustable robust models is to limit the relationship structure to a linear control mechanism. In other words, with this approach the future decisions can only be an affine function of random factors. In our setting this would imply

$$u_t = \bar{u}_t + \sum_{i=1}^{t-1} v_i d_i \quad \forall t = 1, \dots, T. \tag{113}$$

Affinely adjustable robust models are developed specifically for linear models and to use them for our purposes we need to apply them to the conservative model (111). Due to linear relationship with u_t and p_t , the latter should be expressed as a function of random demand factors as well. Letting

$$p_t = \bar{p}_t + \sum_{i=1}^{t-1} w_i d_i \quad \forall t = 1, \dots, T, \quad (114)$$

and assuming that for $t = 1$ we have $u_t = \bar{u}_t$ and $p_t = \bar{p}_t$, the conservative linear model (111) can be reformulated as follows

$$\begin{aligned} \min \quad & C \\ \text{s.t.} \quad & C \geq \sum_{t=1}^T \left(c(\bar{u}_t + \sum_{i=1}^{t-1} v_i d_i) + \bar{p}_t + \sum_{i=1}^{t-1} w_i d_i \right), \\ & \bar{p}_t + \sum_{i=1}^{t-1} w_i d_i \geq h \left(\bar{y}_1 + \sum_{i=1}^t (\bar{u}_i + \sum_{j=1}^{i-1} v_j d_j - d_i) \right) \quad \forall t = 1, \dots, T, \\ & \bar{p}_t + \sum_{i=1}^{t-1} w_i d_i \geq -b \left(\bar{y}_1 + \sum_{i=1}^t (\bar{u}_i + \sum_{j=1}^{i-1} v_j d_j - d_i) \right) \quad \forall t = 1, \dots, T, \\ & \bar{u}_t + \sum_{i=1}^{t-1} v_i d_i \geq 0 \quad \forall t = 1, \dots, T, \end{aligned} \quad (115)$$

where all constraints are satisfied for all $(d_1, \dots, d_T) \in \mathcal{D}$, and \mathcal{D} is as defined in (110). The affinely adjustable robust counterpart (115) is a semi-infinite linear problem. It is well established [13] that the tractability of this formulation depends on efficient separation of \mathcal{D} . In particular, when \mathcal{D} is polyhedral the problem (115) is equivalent to a linear problem that can be solved efficiently.

Note that the optimality of base-stock policy implies a piecewise affine relationship between the future decisions and random factors. This property renders affinely adjustable models very compelling for the setting we consider. Moreover, Bertsimas *et al.* [15] demonstrated that affine decision rules are optimal for the linear approximation we consider for the independent uncertainty sets case.

In the next section we test our proposed solution against these three alternative methods presented.

6.7 Computational results

Consider a five period single item inventory problem. The inventory manager plans to utilize the dynamic robust model together with the budget of uncertainty approach. The cost parameters are given as $c = 10$, $b = 12$, and $h = 4$ for every period. It is estimated that the mean demand value $\bar{d}_t = 50$ and maximum demand deviation $\delta_t = 20$ for all $t = 1, \dots, 5$. The uncertainty budget levels are set as $\Gamma = (0.8, 1.5, 2, 2.6, 3.3)$. We assume that the demand at each period is actually normally distributed with mean 50 and standard deviation 10.

We apply our proposed heuristic to this problem and compare the results with the alternatives; static model on a rolling horizon basis, the conservative model by [17] on a rolling horizon basis and affinely adjustable robust model. We have generated 1000 scenarios to test the quality of the solutions. Note that when a particular demand realization is out of the range suggested by the model, we assume that maximum demand deviation occurred and update the budget levels accordingly. The following Table summarizes our results.

Table 2: Comparison of solutions

	Proposed heuristic	Affinely adj. model	Conservative model	Static model
Average cost	2,886	2,855	2,850	2,946
St. Dev.	151	157	193	143
Maximum	3,540	3,574	3,581	3,436
Minimum	2,434	2,392	2,358	2,466

In the same setting, we consider a ten period problem. The cost parameters for this problem are $c = 15$, $b = 20$, and $h = 6$ for every period. Note that the solution technique proposed by [17] for the conservative model assumes fixed cost parameters across periods, our choice on parameters is limited by this assumption. There are no limitations imposed by any of the alternative methods on the uncertainty set parameters. In fact, our heuristic includes an assumption on these parameters to guarantee optimal solution for our modified problem. However, as discussed this assumption is not required to apply the heuristic. Accordingly, here we consider parameters that violate this specific assumption. Let nominal

demand $\bar{d}_t = 40$ and maximum demand deviation $\delta_t = 20$ for all t except for $t = 3$ and $t = 7$. For these two periods assume that we have $\bar{d}_t = 70$ and $\delta_t = 30$. Observe that the condition $\bar{d}_{t+1} \geq 2\delta_t$ does not hold for $t = 3$ and $t = 7$. The uncertainty budget levels are as follows $\Gamma = (0.6, 1.3, 2, 2.6, 3.4, 4.1, 4.9, 5.5, 6.3, 7)$. We assume that the demand at each period is actually normally distributed with mean 60 and standard deviation 20. The following Table summarizes our results for this problem.

Table 3: Comparison of solutions

	Proposed heuristic	Affinely adj. model	Conservative model	Static model
Average cost	8,681	8,585	8,622	8,794
St. Dev.	459	463	596	432
Maximum	10,632	10,734	10,851	10,297
Minimum	7,334	7,172	7,135	7,382

These examples suggest that the alternative approaches provide solutions with different focuses. The static model is the most conservative one in the sense of minimizing variability while sacrificing the average performance. On the other hand the solution to the conservative model seems to be the most aggressive one; providing one of the best results in average cost performance, but with a higher variability than all other alternatives. Our heuristic and affinely adjustable robust model provide comparable and balanced results. Affinely adjustable model seems to perform relatively better if one focuses on average cost. Our heuristic performs decent in terms of average cost while providing a low variability.

CHAPTER VII

CONCLUSION AND FUTURE RESEARCH

7.1 *Conclusion*

In this thesis, we examined different techniques of managing risk in inventory problems. Inventory problems are extensively studied in risk neutral setting. For the needs of risk averse real life inventory managers, these models are not satisfactory. The literature on inventory problems involving risk can be divided into two groups. The first group assumed the complete knowledge of distribution of random factors and tried to use various risk averse objectives. As an alternative, the second group assumed very little information about the distribution and optimized the worst case values. A detailed coverage of research history for both approach can be found in Chapter 2.

In our work we give a unifying treatment of these two streams of research. This is accomplished through the use of coherent risk measures. Moreover, we analyze static and dynamic robust models in multi-period inventory problems. We present various results on optimal policies for dynamic robust models again by employing coherent risk measures.

In Chapter 3, we use the notion of coherent risk measures in the context of single period newsvendor problem. We establish a one-to-one correspondence between risk averse and min-max type formulations. We derive the optimal solution for a single period newsvendor problem where coherent risk measures is used to control cost variability. We also analyze monotonicity properties of the optimal order quantity with respect to the degree of risk aversion for certain risk measures. In Chapter 4, a risk averse model for the multi-period newsvendor problem that use coherent risk measures is build and analyzed. The one-to-one correspondence described above for single period models holds also for multi-period models. The optimality of base-stock policy for such problems is proved. Furthermore,

the conditions for the optimality of a myopic base-stock policy are presented. Our analysis also includes the case when there is a fixed ordering cost.

A result in Chapter 4 provides a min-max type model different than classical ones, in that the formulation is a nested one. In Chapter 5, we examine dynamic and static robust models in the context of multi-period inventory problems. We provide examples demonstrating the shortcomings of classical static robust models in multi-period setting. In particular, we show that it is possible to generate examples where the difference between the optimal value of static models and dynamic model can be arbitrarily large. Furthermore, we provide examples that are not just pathological cases, and illustrate that significant practical value can be gained by utilizing the dynamic model. We formulate dynamic robust models using dynamic programming equations. For the independent uncertainty sets case we prove the optimality of base-stock policy for dynamic robust formulations. This optimality result holds for any convex overage/underage cost function. It turns out that when the overage/underage cost function is piecewise linear, it is possible to show that the dynamic robust formulation is computationally tractable. Moreover, a technique that provides the optimal base-stock levels in closed form when some conditions on problem parameters hold is presented. We also extend our analysis of optimal solution structure to the case of fixed ordering cost and prove the optimality of (s, S) policy.

In Chapter 6, we consider the case of dependent uncertainty sets. We start with the most general version of the problem and formulate the dynamic robust model. We prove that the optimality of base-stock policy carries over to the case of dependent uncertainty sets. However, solving the dynamic robust formulation or analyzing computational complexity for this case is not easy in general. We focus on a specific dependency structure and provide a heuristic solution. Alternative methods to solve the multi-period inventory model when the uncertainty sets are tied with budget of uncertainty approach are limited. These methods are either static in nature, or an approximation of a static model, or an approximation of a conservative version of our original problem. Our approach is based on modifying the

original dynamic model to be able to use our closed form solution from previous chapter. Note that our heuristic can be applied to a general class of parametric dependency structure satisfying a few conditions. We compare our solution against the alternatives and observe that it provides comparable results to static models applied on a rolling horizon basis. In fact, our results are better when deviation on the outcome is also considered.

7.2 *Future research*

Our work concentrates on single-item and single-echelon inventory problems. A natural extension would be to utilize coherent risk measures to formulate multi-item and/or multi-echelon inventory problems and analyzing both single and multi-period versions of these models. In particular for multi-product case, note that when there are no constraints tying the products together, each product can be modeled separately and our results would apply to each individual model. Analyzing the risk averse models and their relationship with robust formulations in the existence of constraints on multiple products, such as a limit on total inventory purchase, would be an interesting extension, especially since this may lead to the application of similar methodology in the context of portfolio optimization.

In our analysis of dynamic robust inventory models, we prove the optimality of base-stock policies. These policies correspond to piecewise linear decision rules. Specifically, one piece of the rule needs to represent not ordering until a certain demand threshold is realized. This type of decision rules can be investigated and compared against affine decision rules.

The budget of uncertainty approach is only one example of dependency structure for which our proposed heuristic can be applied. It is possible to design other parametric dependency structures where our heuristic would be useful. Moreover, for these new structures the only other alternative solution methodology would be affinely adjustable robust model, since the other two alternatives we have considered are developed specifically for budget of uncertainty approach. A direct comparison of our heuristic and affinely adjustable

robust models may provide more insight about the weaknesses and strenghts of both techniques.

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